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Non-integral geometry: Additional term f_A as a regularizing term

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Abstract

In the present paper, we first describe the principal basis of non-integral geometry. Non-integral geometry is a new field of generalized function (distribution) theory, where the effects breaking the symmetry of integration measure have been investigated. In turn, the non-symmetric integration measure (the non-invariant measure) leads to the complex form of the universal, dimension-independent inverse operator with the additional contributions compared to the methods of integral geometry. The additional term with the complex integration measure serves to the extension that improves the image reconstruction procedure. Then, we prove that this additional term f_A in the universal inverse Radon transforms plays a role of the regularizing contribution. In particular, we show that owing to the presence of f_A, the corresponding complex singularities can be eliminated in the image reconstruction process.

Keywords: non-integral geometry, universal form of inversion, image reconstruction problem

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1. Introduction

Integral geometry as a theory studies the symmetry properties of the corresponding measures on the geometrical spaces. In particular, the direct Radon transform (DRT) considered as the singular delta-functional on the well-defined manifold is one of the main objects in integral geometry. Besides, the integral measure constructed by the differential forms in DRT possesses the certain symmetry and leads to the invariant measure [1, 2].

The methods inspired by integral geometry underlie the mathematical tools for computerized tomography (CT), astrophysics, and seismology [3]. Indeed, the inverse Radon transform (IRT) is being used to reconstruct the image of an object under investigation. It is worth noticing that the inversion of Radon transforms meets the problems related to the ill-posedness. It leads to the necessity to regularize the integral representation of the inverse operator (the different methods of regularization can be found, for example, in [4, 5]).

Recently, in [4–7], the regularization of inverse operator has been implemented in the form that ensures the universality of inverse Radon transform. The universality of IRT means the

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independence of inversion on the space dimension. In contrast to [4–7], the old procedure of Radon transform inversion has been realized in the odd and even dimension of space separately and independently of each other. The dependence of old inversion methods on the parity of space dimension can be traced from the use of Courant–Hilbert identities which have the different representations depending on the space dimension [8]. Besides, in the Courant–Hilbert identities, the angular integration has been always performed over the full region of variations, i.e. in the full interval $(0, 2\pi)$.

According to [4–7], the universal representation for IRT can be derived with the help of the standard regularization procedure, used in the generalized function (distribution) theory [1, 2], and without referring the Courant–Hilbert identities.

As demonstrated, if the original homogeneous outset function, which is under DRT, is well localized in the space and does not possess the symmetry properties, the Radon φ-angular dependence receives the definite restrictions, instead of the full region $(0,\, 2\pi)$. Owing to these angular restrictions, the universal IRT involves two contributions f_S and f_A, where the additional term f_A is related to the complex integration measure. This complexity is a very important founding and it opens a possibility to extend and to improve the Tikhonov regularization needed for the different practical applications [5]. In addition, the inhomogeneity of the outset function support leads also to the existence of the additional term f_A independently of the angular restrictions.

All these above-mentioned arguments break the corresponding symmetry that underlies the standard principles of integral geometry and, on the other hand, form the new principles of *non-integral geometry*. Notice that for the practical applications, the proposed methods of non-integral geometry are more suitable and adequate because the original objects corresponding to the outset functions are naturally non-symmetric ones.

In the paper, we outline the basis of non-integral geometry. We demonstrate that non-integral geometry as a new field of functional analysis operates and studies the effects arising from symmetry breaking of the integration measure. Moreover, the non-invariant (non-symmetric) measure results in the complex form of the universal and dimension-independent inversion with the additional contributions. It is important to notice that the additional term with the complex integration measure extends and improves substantively the image reconstruction procedure. According to that, we prove that this additional term f_A in the universal inverse Radon transform plays a role of the regularizing contribution. In particular, we show that owing to the presence of f_A, the corresponding complex singularities can be eliminated in the image reconstruction process [9].

2. Non-integral geometry and universal form of inverse Radon transform

To get started, we provide the basic definitions needed for our study. In order to restore the internal structure described by the given outset function $f(\vec{\mathbf{x}})$ ($\vec{\mathbf{x}} \in \mathbb{R}^n$ for $\forall n$), one has to use the inverse (in our case, in the universal form) Radon transform. As expected, the IRT-operation should express the outset function $f(\vec{\mathbf{x}})$ through the Radon image. To obtain the representation of IRT, we first define the direct Radon transform of $f(\vec{\mathbf{x}})$ by (cf. [1, 2])

$$G \equiv \mathcal{R}[f](\tau, \varphi, \theta_i) \quad \text{with} \quad \mathcal{R}[f](\tau, \varphi, \theta_i) = \int_{\mathbb{R}^n} d\mu_n(\vec{\mathbf{x}})\, f(\vec{\mathbf{x}})\, \delta\big(\tau - \langle \vec{\mathbf{n}}_{\varphi,\theta_i}, \vec{\mathbf{x}}\rangle\big), \tag{2.1}$$

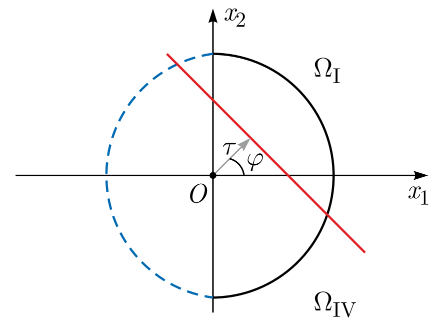

Figure 1. The restricted region in $\vec{\mathbf{x}}$-plane: Ω_{I} and Ω_{IV}. Notations: τ is the Radon radial variable, φ is the Radon angular variable. The line is parametrized as $\tau - \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle = 0$ with $\vec{\mathbf{n}}_\varphi = (\cos\varphi, \sin\varphi)$. For the sake of convenience, the starting point O has been chosen to keep the maximal possible symmetry.

where $\vec{\mathbf{n}}_{\varphi,\theta_i}$ is the n-dimensional unit vector pointing along the radial Radon coordinate. If so, the general form of IRT can be written as

$$f = \mathcal{R}^{-1} G \quad \text{or} \quad f(\vec{\mathbf{x}}) = \int d\mu_n(\eta, \varphi, \theta_i)\, \mathcal{R}[f]\big(\eta + \langle \vec{\mathbf{n}}_{\varphi,\theta_i}, \vec{\mathbf{x}} \rangle, \varphi, \theta_i\big), \tag{2.2}$$

where the integration measure depends now on the radial η and the angular φ, θ_i Radon coordinates (for the demonstration of Radon variables, see Figure 1).

As shown, for example, in [7], the novel approach of Radon inversion does not build upon the Courant–Hilbert identities which make the inversion to be substantially different in the odd and even space dimensions. Instead, we use the Fourier slice theorem that ensures the universality of inversion [1], see Appendix A for details. However, the use of the Fourier slice theorem relates to the necessity of the corresponding regularization owing to the hidden singularity encoded in the Fourier slice theorem, see Appendix A. That is, in our approach, the IRT-operator demands the regularization. After the regularization procedure (see (A.13) of Appendix A), we obtain that

$$f(\vec{\mathbf{x}}) \Longrightarrow f_\varepsilon(\vec{\mathbf{x}}) = \mathcal{R}_\varepsilon^{-1} G \quad \text{and} \quad f_\varepsilon(\vec{\mathbf{x}}) = f_S(\vec{\mathbf{x}}) + f_A(\vec{\mathbf{x}}), \tag{2.3}$$

where $\mathcal{R}_\varepsilon^{-1}$ denotes symbolically the regularized IRT-operator, while $f_S(\vec{\mathbf{x}})$ and $f_A(\vec{\mathbf{x}})$ take the following forms:

$$f_S(\vec{\mathbf{x}}) = i^{\,n-2}(-)^{n-1}(n-1)! \int\limits_{\Omega} d^{n-1}\mu(\varphi, \theta_i) \int\limits_{-\infty}^{+\infty} (d\eta)\, \frac{\mathcal{P}}{\eta^n}\, \mathcal{R}[f](\eta + \langle \vec{\mathbf{n}}_{\varphi,\theta_i}, \vec{\mathbf{x}} \rangle, \varphi, \theta_i) \tag{2.4}$$

and [2]

$$f_A(\vec{\mathbf{x}}) = (-)^{n-1} i^{\,n-1}\, \pi \int\limits_{\Omega} d^{n-1}\mu(\varphi, \theta_i) \int\limits_{-\infty}^{+\infty} (d\eta)\, \delta(\eta)\, \frac{\partial^{n-1}}{\partial \eta^{n-1}}\, \mathcal{R}[f](\eta + \langle \vec{\mathbf{n}}_{\varphi,\theta_i}, \vec{\mathbf{x}} \rangle, \varphi, \theta_i). \tag{2.5}$$

In these equations, $\mathcal{P}$ means the principal value as usual, Ω defines the region of integration with respect to the angular Radon variables. The exact domain of Ω has been specified below.

In the standard theory of integral geometry, there are no angular restrictions, i.e. we deal with the full regions of angular integrations. As a rule, it is because of the certain symmetry

[1] In this case, the universality of inversion means the independency of inversion on the space dimension.

[2] To avoid the existence of surface terms, we here suppose $\mathcal{R}[f](\eta; ...)$ to be a restricted function of η, see [7].

of the outset function support, see below. Moreover, if we believe additionally that the outset function $f(\vec{\mathbf{x}})$ is a homogeneous function with the trivial holonomy (that is, no holonomy at all), then f_S of (2.4) contributes only to the even dimension of space, while f_A of (2.5) — only to the case of the odd dimension. It can be readily demonstrated provided (a) we make the simultaneous replacements: $\eta \to -\eta$, $\vec{\mathbf{n}}_{\varphi,\theta_i} \to -\vec{\mathbf{n}}_{\varphi,\theta_i}$ in (2.4) and (2.5); (b) we take into account the corresponding symmetry properties of η-dependent coefficient functions [3]. It is worth noticing that, in integral geometry, the term of (2.4) contributes only to the even-dimension space, while the term of (2.5) — to the odd-dimension space. Moreover, the contributing term is always real, but the disappearing term is always imaginary.

However, in practice, the reconstructed image corresponding to the original outset function is not symmetric and is inhomogeneous [5–7]. The investigation of symmetry breaking effects together with the inhomogeneity of outset function is a topic for a new branch of theory that can be called as *non-integral geometry*. In this case, the angular integration measure receives the corresponding restrictions dictated by some asymmetry. In addition, the regularized IRT-operator becomes the complex operator (due to the fact that the disappearing, in the standard case, term now does contribute too) which affects on the whole reconstruction procedure.

Thus, the sum of two contributions (see (2.3)–(2.5)) gives the basis for the universal Radon inversion in the complex represention.

3. Direct Radon transform of the probed outset function with the non-symmetric support in $\mathbb{R}^2$

In this section, we present shortly the direct Radon transformation on the chosen function $f(\vec{\mathbf{x}})$ with the simple support.

We begin with the outset function $f(\vec{\mathbf{x}})$, where $\vec{\mathbf{x}} \in \mathbb{R}^2$, defined as (see Figure 1)

$$f(\vec{\mathbf{x}}) = m_{\mathrm{I}}\, \Theta\big(\vec{\mathbf{x}} \in \Omega_{\mathrm{I}}\big) + m_{\mathrm{IV}}\, \Theta\big(\vec{\mathbf{x}} \in \Omega_{\mathrm{IV}}\big). \tag{3.1}$$

In (3.1), the characteristic functions [3] Θ are associated with the supports given by the following regions:

$$\{\Omega_{\mathrm{I}} : x_1 > 0, x_2 > 0, |\vec{\mathbf{x}}| \leqslant 1\} \quad \text{and} \tag{3.2}$$
$$\{\Omega_{\mathrm{IV}} : x_1 > 0, x_2 < 0, |\vec{\mathbf{x}}| \leqslant 1\}. \tag{3.3}$$

The choice of two-dimensional space is mainly dictated by CT-technology, but not only by simplicity. In practice, the different algorithms have been designed for the two-dimensional reconstructions which work with the corresponding transverse projections of the three-dimensional object, see, for example, [5–7].

By the direct calculation of Radon transformation (see Appendix B for details), we obtain that $\mathcal{R}[f](\tau,\varphi)$ contains the non-trivial φ-angular dependence:

$$\mathcal{R}[f](\tau,\varphi) = M\sqrt{1-\tau^2}\,\Theta_1\big(\tau,\,\varphi\big) + M\,\tau\,\cot\varphi\,\Theta_2\big(\tau,\,\varphi\big) - M\,\tau\,\tan\varphi\,\Theta_3\big(\tau,\,\varphi\big), \tag{3.4}$$

where

$$\begin{aligned} &\Theta_1\big(\tau,\,\varphi\big) = \Theta\big(\tau \in [\sin\varphi,\,1]\big) + \Theta\big(\tau \in [0,\,1]\big) + \Theta\big(\tau \in [\cos\varphi,\,1]\big), \\ &\Theta_2\big(\tau,\,\varphi\big) = \Theta\big(\tau \in [0,\,\sin\varphi]\big), \quad \Theta_3\big(\tau,\,\varphi\big) = \Theta\big(\tau \in [\cos\varphi,\,1]\big), \end{aligned} \tag{3.5}$$

and $M = m_{\mathrm{I}} + m_{\mathrm{IV}}$.

[3]The characteristic function equals unit if the condition written as an argument takes place, otherwise it equals zero. It reminds the standard theta-function but without an ambiguity.

Despite the Radon φ-angle varies in (3.4) within the interval $(0,\,\pi/2)$, the expression of (3.4) involves both (3.2) and (3.3) owing to the implemented replacements of variables.

It can be readily seen that if we extend the function support domain up to the full disk covering also $\Omega_{\rm II}+\Omega_{\rm III}$, in addition to (3.2) and (3.3), the Radon φ-angular dependence disappears provided $m_{\rm I}=m_{\rm IV}$. In other words, in the case of the support defined by the symmetric disk, the angular dependence becomes degenerated. It corresponds to the standard fact well known from the literature.

4. Universal form of Radon inversion in $\mathbb{R}^2$

Using (2.4) and (2.5) for $\vec{\mathbf{x}}\in\mathbb{R}^2$, we are going over to the inverse Radon transformation in the universal form. As mentioned, the regularized and universal IRT is given by

$$f_\varepsilon(\vec{\mathbf{x}}) = f_S(\vec{\mathbf{x}}) + f_A(\vec{\mathbf{x}}), \tag{4.1}$$

where $f_S(\vec{\mathbf{x}})$ and $f_A(\vec{\mathbf{x}})$ are the standard and additional contributions, respectively, defined now as

$$f_S(\vec{\mathbf{x}}) = -\int\limits_0^{\pi/2}(d\varphi)\int\limits_{-\infty}^{+\infty}(d\eta)\frac{\mathcal{P}}{\eta^2}\mathcal{R}[f](\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle,\varphi), \tag{4.2}$$

$$f_A(\vec{\mathbf{x}}) = (-i\pi)\int\limits_0^{\pi/2}(d\varphi)\int\limits_{-\infty}^{+\infty}(d\eta)\delta(\eta)\frac{\partial}{\partial\eta}\mathcal{R}[f](\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle,\varphi). \tag{4.3}$$

In (4.2) and (4.3), the special attention should be paid for the origin of complexity. The complex prefactor in (4.3) appears as a result of the corresponding analytical regularization of the inversion procedure, see [5–7] for details. However, as one can see below, the regularization is not the only source of complexity.

Inserting (3.4) into (4.2) and (4.3), we get the following representations:

$$f_S(\vec{\mathbf{x}}) = f_S^{(\tau)}(\vec{\mathbf{x}}) + f_S^{(\varphi)}(\vec{\mathbf{x}}), \tag{4.4}$$

$$f_A(\vec{\mathbf{x}}) = f_A^{(\tau)}(\vec{\mathbf{x}}) + f_A^{(\varphi)}(\vec{\mathbf{x}}), \tag{4.5}$$

where

$$f_S^{(\tau)}(\vec{\mathbf{x}}) = -M\int\limits_0^{\pi/2}(d\varphi)\int\limits_{-\infty}^{+\infty}(d\eta)\frac{\mathcal{P}}{\eta^2}\sqrt{1-\big(\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle\big)^2}\,\Theta_1\big(\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle,\,\varphi\big), \tag{4.6}$$

$$f_S^{(\varphi)}(\vec{\mathbf{x}}) = -M\int\limits_0^{\pi/2}(d\varphi)\int\limits_{-\infty}^{+\infty}(d\eta)\frac{\mathcal{P}}{\eta^2}\Big(\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle\Big)$$
$$\times\Big\{\cot\varphi\,\Theta_2\big(\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle,\,\varphi\big) - \tan\varphi\,\Theta_3\big(\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle,\,\varphi\big)\Big\} \tag{4.7}$$

and

$$f_A^{(\tau)}(\vec{\mathbf{x}}) = (-i\pi)M \int_0^{\pi/2} (d\varphi) \int_{-\infty}^{+\infty} (d\eta)\delta(\eta)\frac{\partial}{\partial\eta}\sqrt{1-\big(\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle\big)^2}\,\Theta_1\big(\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle,\,\varphi\big), \tag{4.8}$$

$$\begin{aligned} f_A^{(\varphi)}(\vec{\mathbf{x}}) &= (-i\pi)M \int_0^{\pi/2} (d\varphi) \int_{-\infty}^{+\infty} (d\eta)\delta(\eta)\frac{\partial}{\partial\eta}\Big(\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle\Big) \\ &\times \Big\{ \cot\varphi\,\Theta_2\big(\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle,\,\varphi\big) - \tan\varphi\,\Theta_3\big(\eta+\langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle,\,\varphi\big)\Big\}. \end{aligned} \tag{4.9}$$

In (4.4) and (4.5), the η-integrations together with other computational reductions of integrations are collected in Appendix.

It is important to notice that the restricted outset function support leads to the non-trivial Radon φ-dependence together with the different sources of complexity in the universal form of inverse Radon transform. Besides, as it is demonstrated by calculations, the extended standard term of universal IRT involving both $f_S^{(\tau)}$ and $f_S^{(\varphi)}$ contains the analytical singularities which are and are not dependent on the position $\vec{\mathbf{x}}$. In this connection, as it turns out, the additional term f_A plays the extremely important role as a regularizer of these singularities, see below.

5. The simplifications in the standard contributions

Before going further, we consider the possible simplification in the standard contributions which involve both the quadratic and linear τ-dependence of DRT. In particular, we describe the cancellation which leads to the simplification of the standard term. As shown in Appendix D, for the tangent-part of the standard contributions, it is convenient to split $f_S^{(\varphi)}(\vec{\mathbf{x}})\big|_{\text{tan-term}}$ into two parts, see (D.4). Then, we have found that the surface terms $J_{\text{s.t.}}$, which are traced from the standard contribution $f_S^{(\tau)}$ (see (C.4) of Appendix C), are cancelled by some of the terms from $f_S^{(\varphi)}$, see (D.14) and (D.15). Indeed, we derive the following relations:

$$\begin{aligned} f_S^{(\tau)}(\vec{\mathbf{x}})\Big|^{A_2}_{\text{surf.term}} &= -M\int_0^{\pi/2}(d\varphi) J_{\text{s.t.}}\big(A_4;\,A_2\big) = -\,f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(b)}_{\text{cot-term}}, \\ f_S^{(\tau)}(\vec{\mathbf{x}})\Big|^{A_2^\star}_{\text{surf.term}} &= -M\int_0^{\pi/2}(d\varphi) J_{\text{s.t.}}\big(A_4;\,A_2^\star\big) = -\,f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,\,(b)}_{\text{tan-term}}. \end{aligned} \tag{5.1}$$

Hence, we finally get that

$$\begin{aligned} f_S^{(\tau)}(\vec{\mathbf{x}})\Big|^{A_2}_{\text{surf.term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(b)}_{\text{cot-term}} &\equiv -f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(b)}_{\text{cot-term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(b)}_{\text{cot-term}} = 0, \\ f_S^{(\tau)}(\vec{\mathbf{x}})\Big|^{A_2^\star}_{\text{surf.term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,\,(b)}_{\text{tan-term}} &\equiv -f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,\,(b)}_{\text{tan-term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,\,(b)}_{\text{tan-term}} = 0. \end{aligned} \tag{5.2}$$

Notice that, depending on the positions in Ω_{I} and Ω_{IV}, such terms as

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(b)}_{\text{cot-term}} \quad \text{and} \quad f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,\,(b)}_{\text{tan-term}}$$

can produce the $\vec{\mathbf{x}}$-dependent singularities in $\Re$e-space. In this sense, the relations (5.2) can be also treated as the regularizing conditions leading to the corresponding singularity cancellations.

6. The complexity of the standard contributions

From (4.4) and (4.5), one can see that the contributions of f_A are accompanied by the complex prefactors [4] $(-i\pi)$ and there are no other complexities in f_A. So, in contrast to f_S, the term f_A is the complex (imaginary) term by construction. However, even the standard contributions of f_S can also generate the definite complexities owing to the cuts of the corresponding logarithmic and polylogarithmic functions.

6.1. $f_S^{(\tau)}$-term

In (C.7), the φ-dependent integrands

$$J^{\text{log.}}_{\text{int.}}\big(A_4;\,-A_1\big),\quad J^{\text{log.}}_{\text{int.}}\big(A_4;\,A_2\big),\quad \text{and}\quad J^{\text{log.}}_{\text{int.}}\big(A_4;\,A_2^\star\big)$$

in $f_S^{(\tau)}$ generate the complexities due to the negative arguments of the corresponding logarithms:

$$\log\big(-A_1\big),\quad \theta\big(-A_2\big)\,\log\big(A_2\big),\quad \text{and}\quad \theta\big(-A_2^\star\big)\,\log\big(A_2^\star\big).$$

We thus have the following terms which correspond to the imaginary parts:

$$f_S^{(\tau)}(\vec{\mathbf{x}})\Big|^{\text{Im.}}_{\text{log.}} = -\,M\int\limits_0^{\pi/2}(d\varphi)\frac{A_1(\varphi)}{\sqrt{1-A_1^2(\varphi)}} \\ \times\Big\{i\,\alpha_1\,\pi\theta\big(A_1(\varphi)\big)+i\,\alpha_2\,\pi\theta\big(-A_2(\varphi)\big)+i\,\alpha_3\,\pi\theta\big(-A_2^\star(\varphi)\big)\Big\}, \tag{6.1}$$

where the uncertainty parameters [5] come from $\log(-1)=i\,\alpha\pi$ with $\alpha=\pm1$. In its turn, the theta-functions of (6.1) give the angular constraints as

$$\theta\big(A_1(\varphi)\big)\Rightarrow\cot\varphi > -\frac{x_2}{x_1}, \tag{6.2}$$

$$\theta\big(-A_2(\varphi)\big)\Rightarrow\cot\varphi > \frac{1-x_2}{x_1}, \tag{6.3}$$

$$\theta\big(-A_2^\star(\varphi)\big)\Rightarrow\cot\varphi > \frac{1-x_1}{x_2}. \tag{6.4}$$

In the Ω_{I}-region of $\vec{\mathbf{x}}$, the inequality (6.2) is always true and, therefore, there is no influence on φ-integration; the inequalities (6.3) and (6.4) give the restrictions

$$\varphi<\arctan\frac{1-x_2}{x_1}\quad\text{and}\quad\varphi>\arctan\frac{1-x_1}{x_2},\quad\text{respectively.} \tag{6.5}$$

In the Ω_{IV}-region of $\vec{\mathbf{x}}$, the inequalities (6.2) and (6.3) reduce to (with $x_2=-\tilde{x}_2$)

$$\varphi<\arctan\frac{\tilde{x}_2}{x_1}\quad\text{and}\quad\varphi<\arctan\frac{1+\tilde{x}_2}{x_1},\quad\text{respectively,} \tag{6.6}$$

while the inequality (6.4) always takes place and there are no other restrictions in this region.

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[4] In general case, the complex prefactor can be associated with the integration measure.

[5] As is well known, $\log(-1)=i\pi(1+2k)$, where $k=0,\pm1,\pm2,\ldots$ For our case, we choose two independent values $k=0$ and $k=-1$. It is reflected in the uncertainty parameter α.

6.2. $f_S^{(\varphi)}$-term

We are now going over to the discussion of the different sources of complexities in $f_S^{(\varphi)}$. To this end, we present $f_S^{(\varphi)}$ as a sum of the structure contributions (see (D.1) and other equations in Appendix D):

$$f_S^{(\varphi)}(\vec{\mathbf{x}}) = f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}} = \Big\{ f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(a)}_{\text{cot-term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(b)}_{\text{cot-term}} \Big\} + \Big\{ f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_1}_{\text{tan-term}} + \Big[f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,(a)}_{\text{tan-term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,(b)}_{\text{tan-term}} \Big] \Big\}, \tag{6.7}$$

where all of the structure contributions are given by (D.12)–(D.15). Hence, after the simplifications considered in Section 5, we leave with $f_S^{(\varphi)}(\vec{\mathbf{x}})$ which can be written as

$$f_S^{(\varphi)}(\vec{\mathbf{x}}) = f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(a)}_{\text{cot-term}} + \Big\{ f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_1}_{\text{tan-term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,(a)}_{\text{tan-term}} \Big\}. \tag{6.8}$$

Since the boundary of Ω_{I} and Ω_{IV} described by the semi-circle $\big\{\vec{\mathbf{x}} \in \Omega_{\text{I,IV}} \;:\; |\vec{\mathbf{x}}| = 1\big\}$ has to be excluded from the consideration [4–7], the logarithmic function of $f_S^{(\varphi)}(\vec{\mathbf{x}})\big|^{N_1}_{\text{tan-term}}$ (see (D.10)) has a negative argument that gives the corresponding complexity. To see that, we can rewrite (D.10) in the form of

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_1}_{\text{tan-term}} = M \int\limits_0^{\pi/2} (d\varphi)\, \tan\varphi \Big[i\alpha\pi + \log\frac{1 - A_1(\varphi)}{A_1(\varphi)} + \frac{A_1(\varphi)}{A_1(\varphi) - 1} - 1 \Big], \tag{6.9}$$

where the condition $A_1(\varphi) < 1$ has been used for the logarithmic argument. Notice that it is one of the reasons for the imaginary terms in the standard contributions. The other reason of the possible complexity in (6.9) is the direct calculation of the integrations. Having used the standard trigonometric replacement of integration variables, the result of calculations in (6.9) has been expressed through the logarithmic functions with the possible negative arguments as well.

The similar situation takes place with the terms $f_S^{(\varphi)}(\vec{\mathbf{x}})\big|^{(a)}_{\text{cot-term}}$ and $f_S^{(\varphi)}(\vec{\mathbf{x}})\big|^{N_2,(a)}_{\text{tan-term}}$. Again, using the trigonometric replacements in the integrals of (D.12) and (D.13) for calculations, we can readily obtain the logarithmic together with polylogarithmic functions which generate the imaginary parts due to the negative arguments of logarithms and the large (greater than unit) argument of polylogarithms.

After some algebra, collecting only the complex terms, we get the following representation:

$$\begin{aligned}
&f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(a)}_{\text{cot-term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,(a)}_{\text{tan-term}} \overset{M}{\sim} \\
&\Big\{ i\alpha_1 \frac{\pi}{4} \log\big(1 + A^2\big) + i\alpha_2 \frac{\pi}{4} \log\big(1 + B^2\big) - i\alpha_3 \pi \log\big(B\big) \Big\} \\
&-\Big\{ \text{the analogous term with replacements: } A \to A^\star;\; B \to B^\star \Big\},
\end{aligned} \tag{6.10}$$

where

$$A = \frac{1 - x_2}{x_1}, \quad B = \frac{x_2}{x_1}, \quad A^\star = \frac{1 - x_1}{x_2}, \quad B^\star = \frac{x_1}{x_2}. \tag{6.11}$$

It is important to stress that in (6.10), we have the complex contributions which are $\vec{\mathbf{x}}$-dependent ones and they can be singular in the definite positions. The cancellations of the possible singularities are discussed in the next section.

To conclude this section, we have demonstrated and discussed all sources of complexities appearing in the standard contributions. Notice that, in contrast to the additional contributions f_A, the standard contributions f_S deal with the real integration measure. However, the complexities appear from the direct calculation of integrals in IRT of the given outset function, see (3.1). Such a property is not because of the special form of the chosen outset function, but it is rather the general property of any IRT in the case of the non-symmetric supports of outset functions.

7. The cancellation of complex singularities: the regularizing role of f_A

In the preceding section, we have discussed the origination of the complex terms in the standard contributions f_S. We remind that, by construction, the additional contributions f_A operate with the imaginary integration measure ensuring the considered complexity and there are no other sources of complexities in f_A. It turned out, in the case of possible complex singularities, the additional term f_A plays the crucial role in the corresponding cancellations (or in the regularization). In this section, our special attention is paid to the cancellation of complex (imaginary) singularities.

It is convenient to begin the study of the regularization procedure with (6.1) that represents the one of complexity sources and is related to $f_S^{(\tau)}$. The φ-integration is given by the typical integral, it reads

$$f_S^{(\tau)}(\vec{\mathbf{x}})\Big|_{\text{log.}}^{\text{Im.}} \sim \int\limits_{C_1}^{C_2} (d\varphi)\frac{A_1(\varphi)}{\sqrt{1-A_1^2(\varphi)}} = \text{Arcsinh}\big(\text{T}\big)\Big|_{C_d}^{C_u}, \tag{7.1}$$

where the integration limits C_1 and C_2 have been defined by the theta-functions in (6.1), and

$$C_d = \frac{r\sin(C_1-\phi)}{\sqrt{1-r^2}}, \quad C_u = \frac{r\sin(C_2-\phi)}{\sqrt{1-r^2}}. \tag{7.2}$$

In (7.1) and (7.2), the polar system for $\vec{\mathbf{x}} = (r,\,\phi)$ has been used as an intermediate step. Further, if the integration limits $C_1 = 0$ and $C_2 = \pi/2$ (this is a result of one of theta-functions), then we derive that

$$f_S^{(\tau)}(\vec{\mathbf{x}})\Big|_{\text{log.}}^{\text{Im.}} \sim \text{Arcsinh}\frac{x_1}{\sqrt{1-|\vec{\mathbf{x}}|^2}} + \text{Arcsinh}\frac{x_2}{\sqrt{1-|\vec{\mathbf{x}}|^2}}. \tag{7.3}$$

On the other hand, in general case, we have that

$$\text{Arcsinh}\big(z\big) = \log\big(z+\sqrt{1+z^2}\big), \quad \text{with} \quad z\in\mathbb{C}. \tag{7.4}$$

Hence, for the points $(x_1,\,x_2)\in\Omega_{\text{I,IV}}$ in positions $(0,\,1)$ and $(1,\,0)$, we deal with the singularities like $\log[\infty]$. Fortunately, the additional term of $f_A^{(\tau)}\big|^{(a)}$ (see (E.1) of Appendix E) gives the same contribution as (6.1) but with the opposite common sign provided

$$\alpha_1 = \alpha_2 = \alpha_3 = 1. \tag{7.5}$$

So, we observe the full cancellation independently of whether the singularities have been generated or not. It can be presented as

$$f_S^{(\tau)}(\vec{\mathbf{x}})\Big|^{\text{Im.}}_{\text{log.}} + f_A^{(\tau)}(\vec{\mathbf{x}})\Big|^{(a)} = 0. \tag{7.6}$$

This is the first type of cancellations.

The next stage is to consider the logarithmic term of $f_S^{(\varphi)}(\vec{\mathbf{x}})\big|^{N_1}_{\text{tan-term}}$ that leads to the $\vec{\mathbf{x}}$-independent logarithmic singularity [6]:

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_1,\log[\infty]}_{\text{tan-term}} = i\,\alpha\pi\,M \int\limits_0^{\pi/2} (d\varphi)\,\tan\varphi \overset{M}{\sim} i\,\alpha\pi\,\log[\infty]. \tag{7.7}$$

This type of singularities is the most dangerous owing to the $\vec{\mathbf{x}}$-independence. However, there are the other terms which can generate the similar singularities in

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(a)}_{\text{cot-term}} \quad \text{and} \quad f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,(a)}_{\text{tan-term}}. \tag{7.8}$$

Besides, the singularities produced in the terms of (7.8) depend on the positions in $\vec{\mathbf{x}}$-space. We call them $\vec{\mathbf{x}}$-dependent singularities. In particular, at the points given by (0, 1) and (1, 0), we have the following combination:

$$\begin{aligned}
&f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(a),\log[\infty]}_{\text{cot-term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,(a),\log[\infty]}_{\text{tan-term}} \overset{M}{\sim} \\
&\Big\{ - i\tilde{\alpha}_1 \frac{\pi}{2}\log[\infty] + i\tilde{\alpha}_2\pi\log[\infty] + i\tilde{\alpha}_3\pi\log[\infty]\Big\}_{\text{at }(0,\,1)} \\
&+ \Big\{ - i\tilde{\beta}_1\pi\log[\infty] + i\tilde{\beta}_2\frac{\pi}{2}\log[\infty] - i\tilde{\beta}_3\pi\log[\infty]\Big\}_{\text{at }(1,\,0)}
\end{aligned} \tag{7.9}$$

and

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_1,\log[\infty],\vec{\mathbf{x}}\text{-dep.}}_{\text{tan-term}} \overset{M}{\sim} \Big\{ - i\alpha_5\pi\log[\infty]\Big\}_{\text{at }(1,\,0)}. \tag{7.10}$$

Further, we dwell on the additional f_A-contributions which can generate the singularities. We find that the term $f_A^{(\tau)}(\vec{\mathbf{x}})\big|^{(b)}$ (see (E.2)) at the points (0, 1) and (1, 0) gives the following singularities:

$$f_A^{(\tau)}(\vec{\mathbf{x}})\Big|^{(b),\log[\infty]} \overset{M}{\sim} \Big\{ - i\frac{\pi}{2}\log[\infty]\Big\}_{\text{at }(0,\,1)} + \Big\{ - i\frac{\pi}{2}\log[\infty]\Big\}_{\text{at }(1,\,0)} = -i\pi\log[\infty], \tag{7.11}$$

while the additional term $f_A^{(\varphi)}$ (see (F.1)–(F.3) of Appendix F) at the points (0, 1) and (1, 0) leads to the following singularities:

$$f_A^{(\varphi)}(\vec{\mathbf{x}})\Big|^{\log[\infty]}_{\text{cot-term}} = \Big\{ i\frac{\pi}{2}\log[\infty]\Big\}_{\text{at }(0,\,1)} \tag{7.12}$$

and

$$f_A^{(\varphi)}(\vec{\mathbf{x}})\Big|^{\log[\infty]}_{\text{tan-term}} = \Big\{ i\pi\log[\infty]\Big\}_{\text{at }(0,\,1)} + \Big\{ i\pi\log[\infty] + i\frac{\pi}{2}\log[\infty]\Big\}_{\text{at }(1,\,0)}. \tag{7.13}$$

[6] Here and below, we assume that $\log[\infty] \equiv \lim_{t\to\infty}\log(t)$.

Summarizing all terms with the singularities presented by (7.7)–(7.13), we obtain that the cancellation of singularities takes place provided

$$\begin{aligned} &\tilde{\alpha}_2 = \tilde{\beta}_1 = 1, \quad \tilde{\alpha}_3 = \tilde{\beta}_3 = 1, \\ &\tilde{\alpha}_1 = \tilde{\beta}_2 = 1, \quad \alpha_5 = -\alpha = 1. \end{aligned} \tag{7.14}$$

Notice that this set of parameters is not unique, we are able to arrange the other two sets in addition:

$$\begin{aligned} &\tilde{\alpha}_2 = -\tilde{\beta}_1 = -1, \quad \tilde{\alpha}_3 = -\tilde{\beta}_3 = 1, \\ &\tilde{\alpha}_1 = \tilde{\beta}_2 = 1, \qquad \alpha_5 = -\alpha = 1 \end{aligned} \tag{7.15}$$

and

$$\begin{aligned} &\tilde{\alpha}_2 = -\tilde{\beta}_1 = 1, \quad -\tilde{\alpha}_3 = \tilde{\beta}_3 = 1, \\ &\tilde{\alpha}_1 = \tilde{\beta}_2 = 1, \qquad \alpha_5 = -\alpha = 1. \end{aligned} \tag{7.16}$$

All three sets of parameters (7.14)–(7.16) are independent, and they can be equivalently used. Symbolically, the second type of cancellations can be presented as

$$\begin{aligned} f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_1,\log[\infty]} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_1,\log[\infty],\vec{\mathbf{x}}\text{-dep.}} + \Big\{ f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}}^{(a),\log[\infty]} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_2,(a),\log[\infty]} \Big\} \\ + f_A^{(\tau)}(\vec{\mathbf{x}})\Big|^{(b),\log[\infty]} + \Big\{ f_A^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}}^{\log[\infty]} + f_A^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{\log[\infty]} \Big\} = 0. \end{aligned} \tag{7.17}$$

Thus, two types of the singularity cancellations given by (7.6) and (7.17) present the main result of the paper. We can see that the additional contribution f_A with the complex integration measure plays the important role of the regularizing term.

8. Conclusions

In conclusion, we have presented the basis of non-integral geometry which can be treated as a new field of functional analysis. In order to clarify the main differences in comparison with the standard integral geometry, it is worth mentioning the following matching. The standard methods of integral geometry are based on:

- Courant–Hilbert identities traced from Green's formulae. It gives the different representations of inversion for the spaces with even and odd dimensions;
- invariant (symmetric) integration measure that leads to the full integration region: $\varphi \in (0, 2\pi)$.

On the other hand, the proposed methods of non-integral geometry are built upon the following stages:

- exception of Courant–Hilbert identities (Green's formulae) from the consideration. Instead, we use the Fourier slice theorem that demands the corresponding regularization of inversion. As a result, we have derived the universal representation of inversion for the spaces independently of even and odd dimensions;
- non-invariant (non-symmetric) integration measure that leads to the restricted integration region, for example, $\varphi \in (-\pi/2, \pi/2)$.

Inspired by non-integral geometry, we have studied the effects of symmetry breaking related to the integration measure. As demonstrated, the non-invariant (non-symmetric) measure results directly in the complexity of the universal inversion with the additional contributions.

In the presented paper, we have advocated a unique role of the additional term f_A. Namely, the term f_A is very important to regularize the universal IRT of the outset function located in some domain. Despite we have considered the simplest form of the outset function, the discovered role of the additional term as the regularizing contribution is the very general property even for the practical outset function. We have emphasized that the mentioned regularization is very important for the image reconstruction procedure [9].

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Conflicts of interest

The author declares no conflicts of interest.

Data Availability Statement

This manuscript has no associated data or the data will not be deposited.

Appendix

In this Appendix, we present the needed technical details of calculations and the results of analytical calculations of f_S and f_A.

Appendix A. Fourier slice theorem: axis singularity vs. unboundedness

We are now in a position to discuss in detail the fundamental theorem called the *Fourier slice theorem*. As shown in a series of papers [4–7], this theorem plays the principle role in the derivation of universal Radon inversion.

In particular, we demonstrate that the unboundedness of Radon transforms and the axis singularities, well known in physics, are related to each other. This fact demands the corresponding regularization in a sense of generalized function theory.

First, we consider the well-defined Fourier transform [7] of some regular function $f(\vec{\mathbf{x}})$ with $\vec{\mathbf{x}} \in \mathbb{R}^2$:

$$\mathcal{F}[f](\vec{\mathbf{q}}) = \int\limits_{-\infty}^{+\infty} (d^2\vec{\mathbf{x}})\, \mathrm{e}^{-i\langle \vec{\mathbf{q}}, \vec{\mathbf{x}}\rangle}\, f(\vec{\mathbf{x}}). \tag{A.1}$$

In this representation, the Fourier coordinates $(q_1,\, q_2)$ and the Cartesian coordinates $(x_1,\, x_2)$ are independent. Notice that the inverse Fourier transforms are also well-defined.

[7] This transform involves the infinite integration limits. The case of the finite integration limits related to the periodic function is beyond the scope of our study due to the fact that they can be matched with the help of the corresponding replacement.

Let us define the straightforward line which can be parametrized by two ways:

$$t - \langle \vec{\mathbf{q}}, \vec{\mathbf{x}} \rangle \equiv t - q_1 x_1 - q_2 x_2 = 0, \tag{A.2}$$

$$z - x_1 - \xi x_2 = 0. \tag{A.3}$$

In (A.1), the Fourier and Cartesian coordinates are still independent of each other despite the imposed condition (A.2). The parametrization of (A.2) has three parameters: $\{t, q_1, q_2\}$, while that of (A.3) — two parameters: $\{z, \xi\}$. Notice that the line parametrization (A.2) is redundant compared to the line parametrization (A.3). Indeed, any line in $\mathbb{R}^2$, which is not coinciding (or parallelizing) with the corresponding axes, has only two points of axis interceptions:

$$\begin{cases} (0,\, t/q_2) \;\text{ and }\; (t/q_1,\, 0) & \text{for (A.2)}, \\ (0,\, z/\xi) \;\text{ and }\; (z,\, 0) & \text{for (A.3)}. \end{cases} \tag{A.4}$$

Hence, it is enough to have only two external parameters to parametrize properly the straightforward line.

Inserting the line parametrization condition (A.2) in the form of

$$1 = \int\limits_{-\infty}^{+\infty} (dt)\, \delta\big(t - q_1 x_1 - q_2 x_2\big), \tag{A.5}$$

we get that

$$\mathcal{F}[f](q_1, q_2) = \int\limits_{-\infty}^{+\infty} (d^2\vec{\mathbf{x}})\, \mathrm{e}^{-i\langle \vec{\mathbf{q}}, \vec{\mathbf{x}}\rangle}\, f(x_1, x_2) \Big\{ \int\limits_{-\infty}^{+\infty} (dt)\, \delta\big(t - q_1 x_1 - q_2 x_2\big) \Big\} \tag{A.6}$$

$$= \int\limits_{-\infty}^{+\infty} (dt)\, \mathrm{e}^{-it} \Big\{ \int\limits_{-\infty}^{+\infty} (d^2\vec{\mathbf{x}})\, f(x_1, x_2) \delta\big(t - q_1 x_1 - q_2 x_2\big) \Big\}. \tag{A.7}$$

It is important to stress that in (A.7), the expression in the curly brackets is not yet giving the direct Radon transformation of $f(x_1, x_2)$ because it depends on three parameters: $\{t, q_1, q_2\}$ instead of needed two [8].

In order to get the direct Radon transformation in (A.6), we introduce one external condition: $\xi = q_2/q_1$. Practically, it means that one singles out the line $q_2 = \xi q_1$ in (q_1, q_2)-plane with a slope parameter ξ. We thus have the following:

$$\mathcal{F}[f](q_1, q_2) \quad \xrightarrow{\xi = q_2/q_1} \quad \mathcal{F}[f](q_1, \xi q_1), \tag{A.8}$$

where in the r.h.s., a new set of two independent parameters: $\{q_1, \xi\}$ is presented.

[8] This statement is known as the following theorem: if f is a function of n independent variables, then the Radon transform of f, $\mathcal{R}[f]$, depends on n independent variables too. The Radon transform is a bijection and lives on $\mathbb{R}^1 \times \mathrm{S}^{n-1}$, see, for example, [3].

Hence, the representation (A.6) is being changed on the following:

$$\mathcal{F}[f](q_1, \xi\, q_1) = \int_{-\infty}^{+\infty} (dt)\, \mathrm{e}^{-it} \Big\{ \frac{1}{|q_1|} \int_{-\infty}^{+\infty} (d^2\vec{\mathbf{x}})\, f(x_1, x_2) \delta\big(t/q_1 - x_1 - \xi x_2\big) \Big\}$$
$$= \int_{-\infty}^{+\infty} (dz)\, \mathrm{e}^{-izq_1} \Big\{ \int_{-\infty}^{+\infty} (d^2\vec{\mathbf{x}})\, f(x_1, x_2) \delta\big(z - x_1 - \xi x_2\big) \Big\} \equiv \int_{-\infty}^{+\infty} (dz)\, \mathrm{e}^{-izq_1}\, \mathcal{R}[f](z, \xi), \quad \text{(A.9)}$$

where the replacement: $z = t/q_1$ has been implemented, and the direct Radon transform of $f(x_1, x_2)$, denoted as $\mathcal{R}[f](z, \xi)$, has been extracted.

Thus, the Fourier slice theorem states that *if the Fourier coordinates* (q_1, q_2) *transform to a system with the new coordinates* (q_1, ξ)*, where* $\xi = q_2/q_1$*, then the new Fourier image of* $f(x_1, x_1)$ *relates to the Radon image of* $f(x_1, x_2)$ *as*

$$\mathcal{F}[f](q_1, \xi\, q_1) = \int_{-\infty}^{+\infty} (dz)\, \mathrm{e}^{-izq_1}\, \mathcal{R}[f](z, \xi), \quad \text{(A.10)}$$

where the coordinates q_1 *and* z *are Fourier-conjugated, while the* ξ*-coordinate for* $\mathcal{F}[f](q_1, \xi\, q_1)$ *and* $\mathcal{R}[f](z, \xi)$ *remains identical.*

In other words, one may claim that the Fourier image of the outset function $f(x_1, x_2)$ calculated along a line with the slope parameter ξ is Fourier-conjugated with the Radon image of the same function $f(x_1, x_2)$ calculated along the set of lines with the same slope parameter ξ. Moreover, thanks for the Fourier slice theorem, the Fourier angular restrictions coincide with the Radon angular restrictions. This advantage has been successfully used in [4–7].

The system of $(q_1, \xi) \equiv (q_1, \xi q_1)$ reminds the polar system for $\vec{\mathbf{q}}$ but it is not identical to that. Indeed, in the $(q_1, \xi q_1)$-system, if one of coordinates is zero, $q_1 = 0$, then the other is nullified too, $q_2 \equiv \xi q_1 = 0$, provided ξ is finite. This property of system also takes place for the polar system of $\vec{\mathbf{q}}$: if $q_1 = \lambda \cos\varphi$ and $q_2 = \lambda \sin\varphi$, then the nullification of both q_1 and q_2 has been ensured by the only requirement: $\lambda = 0$. The nullification of the radial polar coordinate λ leads to the degeneration of φ-dependence. This effect is called the *axis singularity*.

On the other hand, if $q_1 \to 0$, and therefore $q_2 \to 0$, then $\{z \to \infty, |t| \leqslant M < \infty\}$. Thus, $\mathcal{R}[f](z, \xi)$ might be restricted with respect of z-variables to ensure the convergency of integral, see (A.10). However, as is well known, the Radon image $\mathcal{R}[f](z, \xi)$ of any well-localized outset function $f(x_1, x_2)$ is unbounded. This contradiction shows us that the axial singularity at $\lambda \to 0$ (or $q_1 \to 0$) and the unboundedness of Radon image at $z \to \infty$ have the similar nature and they demand the corresponding regularizations similarly.

Saying that, we go over to the polar system for $\vec{\mathbf{q}}$. For this system, using the inverse Fourier transform, we can write down that

$$f(\vec{\mathbf{x}}) = \int_{-\infty}^{+\infty} d^2\vec{\mathbf{q}}\, \mathrm{e}^{+i\langle \vec{\mathbf{q}}, \vec{\mathbf{x}}\rangle}\, \mathcal{F}[f](\vec{\mathbf{q}})\Big|_{\vec{\mathbf{q}} = \lambda \vec{\mathbf{n}}_\varphi} = \int_{0}^{+\infty} d\lambda\lambda \int d\mu(\varphi)\, \mathrm{e}^{+i\lambda\langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}}\rangle}\, \mathcal{F}[f](\lambda, \varphi). \quad \text{(A.11)}$$

Then, we use the Fourier slice theorem (A.10) to get the following [9]:

$$f_\varepsilon(\vec{\mathbf{x}}) = \int d\mu(\varphi) \int\limits_{-\infty}^{+\infty} (d\eta)\, \mathcal{R}[f](\eta + \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}}\rangle, \varphi) \int\limits_{0}^{+\infty} d\lambda\lambda\, \mathrm{e}^{-i\lambda\eta}\Big|_{\varepsilon\text{-reg.}}, \tag{A.12}$$

where "ε-reg." denotes the necessary regularization (see below) and the integration limits for φ-angular variable are irrelevant at this moment. From (A.12), one can see that the integration over λ has been factorized in the part which demands the corresponding regularization. This is the analogous effect of axis (unboundedness) singularity described above, but it manifests in terms of the Radon η-radial variable at $\eta \to 0$. For the ε-regularization, we do $\eta \to \eta - i\varepsilon$ which provides the analytical regularization [1]. In the most general case, the λ-integration reads

$$i^{\,2-n} \int\limits_{0}^{+\infty} d\lambda\lambda^{n-1}\, \mathrm{e}^{-i\lambda(\eta - i\varepsilon)} = i\, \frac{\partial^{n-1}}{\partial\eta^{n-1}} \int\limits_{0}^{+\infty} d\lambda\, \mathrm{e}^{-i\lambda(\eta-i\varepsilon)} = (-)^{n-1}(n-1)!\,\frac{\mathcal{P}}{\eta^n} + i\pi \frac{\partial^{n-1}}{\partial\eta^{n-1}}\delta(\eta). \tag{A.13}$$

One can see that due to (A.13), the inverse Radon representation of (A.12) becomes well-defined.

Appendix B. DRT of the probe outset function

Here, we present the details of the direct Radon transformation which acts on the probe outset function given by (3.1).

Inserting (3.1) into (2.1), we obtain the following (here, $\vec{\mathbf{x}} = (x_1,\, x_2)$):

$$\mathcal{R}[f](\tau, \varphi) = \int\limits_{\Omega_{\mathrm{I}}+\Omega_{\mathrm{IV}}} d^2\vec{\mathbf{x}}\, f(\vec{\mathbf{x}})\, \delta\big(\tau - x_1 \cos\varphi - x_2 \sin\varphi\big) \tag{B.1}$$

$$= \int\limits_{-\pi/2}^{\pi/2} d\phi \int\limits_{0}^{1} dr\, r\, f(r\cos\phi, r\sin\phi)\, \frac{1}{|\cos(\phi-\varphi)|}\delta\Big(\frac{\tau}{\cos(\phi-\varphi)} - r\Big) \tag{B.2}$$

$$= m_{\mathrm{I}}\, \tau \int\limits_{0}^{\pi/2} d\phi\, \frac{\Theta\Big(\frac{\tau}{\cos(\phi-\varphi)} \in [0,\, 1]\Big)}{\cos^2(\phi-\varphi)} + m_{\mathrm{IV}}\, \tau \int\limits_{0}^{\pi/2} d\tilde{\phi}\, \frac{\Theta\Big(\frac{\tau}{\cos(\tilde{\phi}+\varphi)} \in [0,\, 1]\Big)}{\cos^2(\tilde{\phi}+\varphi)}, \tag{B.3}$$

where the polar system for $\vec{\mathbf{x}}$ has been used in (B.2) and the replacement: $\phi = -\tilde{\phi}$ has been done in the second term (with m_{IV}) of (B.3).

It is worth stressing that, as is well known, the integration with delta-function results in the set of conditions encoded in the corresponding characteristic Θ-functions, see (B.3).

The next stage is to disentangle all conditions presented by Θ-functions. For the sake of simplicity, we first assume that the external φ-variable belongs to the interval $(0,\, \pi/2)$. The extension to the full region, $\varphi \in (-\pi/2, \pi/2)$ is trivial, see below. So, we begin with the first term of (B.3) which is proportional to m_{I}. We have

$$\tau \geqslant 0, \tag{B.4}$$

$$\cos(\phi - \varphi) \geqslant 0, \tag{B.5}$$

$$\tau \leqslant \cos(\phi - \varphi). \tag{B.6}$$

[9] ε as a subscript of f denotes the ε-regularization that should be used in (A.12).

The condition (B.4) is trivial, by construction, and it can be omitted in our further discussion unless it may lead to misunderstanding. The condition (B.5) leads to the restrictions, applied to the integration ϕ-variable, given by

$$\Big\{ I_b : \quad -\pi/2 + \varphi \leqslant \phi \leqslant \pi/2 + \varphi \Big\}. \tag{B.7}$$

In addition, we remind that the integration limits of the considered term are $(0, \pi/2)$. Hence, the condition (B.5) is simply reduced to the integration interval

$$I_b \,\cap\, [0,\, \pi/2] = [0,\, \pi/2], \tag{B.8}$$

while the condition (B.6) gives the following restrictions:

$$\Big\{ I_c : \quad -\varphi_\tau + \varphi \leqslant \phi \leqslant \varphi_\tau + \varphi \Big\} \quad \text{with} \quad \arccos\tau = \pm\varphi_\tau \tag{B.9}$$

and it requires more detailed consideration. Indeed, we can readily derive that

$$I_c \,\cap\, [0,\, \pi/2] = \{\text{a set of conditions}\}. \tag{B.10}$$

In (B.10), a set of conditions involves the following restrictions on the ϕ-integration limits and the corresponding conditions for the external parameters τ and φ:

$$\begin{aligned} &\phi \in [0,\, \pi/2] \quad \text{if} \begin{cases} \varphi_\tau + \varphi \geqslant \pi/2 \\ -\varphi_\tau + \varphi \leqslant 0 \end{cases} \Rightarrow \begin{cases} \tau \leqslant \sin\varphi \\ \tau \leqslant \cos\varphi \end{cases} \\ &\Rightarrow \begin{cases} \text{if } \varphi \leqslant \pi/4, & \text{then } \tau \leqslant \sin\varphi \leqslant \cos\varphi \\ \text{if } \varphi \geqslant \pi/4, & \text{then } \tau \leqslant \cos\varphi \leqslant \sin\varphi \end{cases}, \end{aligned} \tag{B.11}$$

and

$$\begin{aligned} &\phi \in [0,\, \varphi_\tau + \varphi] \quad \text{if} \begin{cases} \varphi_\tau + \varphi \leqslant \pi/2 \\ -\varphi_\tau + \varphi \leqslant 0 \end{cases} \Rightarrow \begin{cases} \tau \geqslant \sin\varphi \\ \tau \leqslant \cos\varphi \end{cases} \\ &\Rightarrow \Big\{ \sin\varphi \leqslant \tau \leqslant \cos\varphi, \ \text{ iff } \ \varphi \leqslant \pi/4 \Big\}, \end{aligned} \tag{B.12}$$

and

$$\begin{aligned} &\phi \in [-\varphi_\tau + \varphi,\, \varphi_\tau + \varphi] \quad \text{if} \begin{cases} \varphi_\tau + \varphi \leqslant \pi/2 \\ -\varphi_\tau + \varphi \geqslant 0 \end{cases} \Rightarrow \begin{cases} \tau \geqslant \sin\varphi \\ \tau \geqslant \cos\varphi \end{cases} \\ &\Rightarrow \begin{cases} \text{if } \varphi \leqslant \pi/4, & \text{then } \tau \geqslant \cos\varphi \geqslant \sin\varphi \\ \text{if } \varphi \geqslant \pi/4, & \text{then } \tau \geqslant \sin\varphi \geqslant \cos\varphi \end{cases}, \end{aligned} \tag{B.13}$$

and

$$\begin{aligned} &\phi \in [-\varphi_\tau + \varphi,\, \pi/2] \quad \text{if} \begin{cases} \varphi_\tau + \varphi \geqslant \pi/2 \\ -\varphi_\tau + \varphi \geqslant 0 \end{cases} \Rightarrow \begin{cases} \tau \leqslant \sin\varphi \\ \tau \geqslant \cos\varphi \end{cases} \\ &\Rightarrow \Big\{ \cos\varphi \leqslant \tau \leqslant \sin\varphi, \ \text{ iff } \ \varphi \geqslant \pi/4 \Big\}. \end{aligned} \tag{B.14}$$

Further, we dwell on the second term of (B.3) which is proportional to $m_{\rm IV}$. We write

$$\tau \geqslant 0, \tag{B.15}$$
$$\cos(\tilde\phi+\varphi) \geqslant 0, \tag{B.16}$$
$$\tau \leqslant \cos(\tilde\phi+\varphi). \tag{B.17}$$

As above, the condition (B.15) is trivial and it can be omitted. The condition (B.16) corresponds to the restrictions

$$\Big\{ J_b: \quad -\pi/2-\varphi \leqslant \tilde\phi \leqslant \pi/2-\varphi \Big\}. \tag{B.18}$$

Taking into account that the integration limits of the considered term are also $[0,\,\pi/2]$, the condition (B.16) modifies the integration interval for $\tilde\phi$ as

$$J_b \,\cap\, [0,\,\pi/2] = [0,\,\pi/2-\varphi]. \tag{B.19}$$

In contrast to the first term of (B.3), where the condition (B.7) does not produce any additional constraints, the condition (B.18) does produce the constraint, see (B.19).

Going over to the condition (B.17), one can see that the following restrictions take place:

$$\Big\{ J_c: \quad -\varphi_\tau-\varphi \leqslant \tilde\phi \leqslant \varphi_\tau-\varphi \Big\}. \tag{B.20}$$

Moreover, these constraints are being interfered with (B.19) giving the interval for the $\tilde\phi$-variable:

$$J_c \,\cap\, [0,\,\pi/2-\varphi] = [0,\,\varphi_\tau-\varphi] \tag{B.21}$$

provided (B.15) and

$$-\varphi_\tau-\varphi\leqslant 0 \quad \Rightarrow \begin{cases} \tau\leqslant\cos\varphi, & \text{if } \,|\varphi_\tau|\geqslant|\varphi| \\ \tau\geqslant\cos\varphi, & \text{if } \,|\varphi_\tau|\leqslant|\varphi| \end{cases} \Rightarrow \big\{0\leqslant\tau\leqslant 1\big\}. \tag{B.22}$$

From this, one can see that the integration limits for $\tilde\phi$ given by (B.21) correspond to the natural conditions (B.22) only.

The final stage is to summarize all above-considered contributions with the appropriate conditions. After the corresponding integrations with respect to ϕ- and $\tilde\phi$-variables (see (B.3)), we derive that

$$\begin{aligned}
&\mathcal{R}[f](\tau,\varphi)\Big|^{\varphi\in(0,\,\pi/2)} \\
&= \Big\{\Theta(\varphi\leqslant\frac{\pi}{4})\Theta(\tau\leqslant\sin\varphi\leqslant\cos\varphi)+\Theta(\varphi\geqslant\frac{\pi}{4})\Theta(\tau\leqslant\cos\varphi\leqslant\sin\varphi)\Big\} m_{\rm I}\,\tau\big[\cot\varphi+\tan\varphi\big] \\
&+\Big\{\Theta(\varphi\leqslant\frac{\pi}{4})\Theta(\sin\varphi\leqslant\tau\leqslant\cos\varphi)+\Big\} m_{\rm I}\,\tau\Big[\frac{\sqrt{1-\tau^2}}{\tau}+\tan\varphi\Big] \\
&+\Big\{\Theta(\varphi\leqslant\frac{\pi}{4})\Theta(\tau\geqslant\cos\varphi\geqslant\sin\varphi)+\Theta(\varphi\geqslant\frac{\pi}{4})\Theta(\tau\geqslant\sin\varphi\geqslant\cos\varphi)\Big\} m_{\rm I}\,2\sqrt{1-\tau^2} \\
&+\Big\{\Theta(\varphi\geqslant\frac{\pi}{4})\Theta(\cos\varphi\leqslant\tau\leqslant\sin\varphi)+\Big\} m_{\rm I}\,\tau\Big[\frac{\sqrt{1-\tau^2}}{\tau}+\cot\varphi\Big] \\
&+\Big\{\Theta(0\leqslant\tau\leqslant1)+\Big\} m_{\rm IV}\,\tau\Big[\frac{\sqrt{1-\tau^2}}{\tau}-\tan\varphi\Big].
\end{aligned} \tag{B.23}$$

The inclusion of $\varphi \in (-\pi/2,\, 0)$ in our consideration corresponds to the formal addition of the contribution (B.23), where the replacements: $m_{\rm I} \to m_{\rm IV}$ and $m_{\rm IV} \to m_{\rm I}$ have been implemented.

Finally, after some algebra, we get

$$\mathcal{R}[f](\tau,\varphi) = M\,\sqrt{1-\tau^2}\Big\{\Theta\big(\tau\in[\sin\varphi,\,1]\big)+\Theta\big(\tau\in[0,\,1]\big)+\Theta\big(\tau\in[\cos\varphi,\,1]\big)\Big\} + M\,\tau\,\cot\varphi\Big\{\Theta\big(\tau\in[0,\,\sin\varphi]\big)\Big\} - M\,\tau\,\tan\varphi\Big\{\Theta\big(\tau\in[\cos\varphi,\,1]\big)\Big\}, \quad \text{(B.24)}$$

where $M = m_{\rm I} + m_{\rm IV}$. This expression coincides with (3.4).

Appendix C. The standard contributions with the quadratic τ-dependence of DRT

The standard contributions with the quadratic τ-dependence of DRT are

$$f^{(\tau)}_S(\vec{\mathbf{x}}) = -\,M\int\limits_0^{\pi/2}(d\varphi)\Big\{\int\limits_{-A_1(\varphi)}^{A_4(\varphi)}(d\eta)\frac{\sqrt{1-\big(\eta+A_1(\varphi)\big)^2}}{\eta^2} + \int\limits_{A_2(\varphi)}^{A_4(\varphi)}(d\eta)\frac{\sqrt{1-\big(\eta+A_1(\varphi)\big)^2}}{\eta^2} + \int\limits_{A^\star_2(\varphi)}^{A_4(\varphi)}(d\eta)\frac{\sqrt{1-\big(\eta+A_1(\varphi)\big)^2}}{\eta^2}\Big\}, \quad \text{(C.1)}$$

where the shortened notations have been introduced as (here and below, $\vec{\mathbf{x}}$-dependence is hidden in all A-functions)

$$\begin{aligned} &A_1(\varphi) \equiv \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}}\rangle = x_2\,\sin\varphi + x_1\,\cos\varphi, \quad && A_2(\varphi) = (1-x_2)\,\sin\varphi - x_1\,\cos\varphi,\\ &A^\star_2(\varphi) = (1-x_1)\,\cos\varphi - x_2\,\sin\varphi, && A_4(\varphi) = 1 - A_1(\varphi). \end{aligned} \quad \text{(C.2)}$$

Thanks for the integration by part, the typical η-integration in (C.1) can be presented as

$$J = \int\limits_{A_d(\varphi)}^{A_u(\varphi)}(d\eta)\frac{\sqrt{1-\big(\eta+A_1(\varphi)\big)^2}}{\eta^2} = J_{\rm s.t.}\big(A_d;\,A_u\big) + J_{\rm int.}\big(A_d;\,A_u\big), \quad \text{(C.3)}$$

where the surface term $J_{\rm s.t.}$ and the integration term $J_{\rm int.}$ read

$$J_{\rm s.t.}\big(A_d;\,A_u\big) = -\,\frac{\sqrt{1-\big(\eta+A_1(\varphi)\big)^2}}{\eta}\Big|^{A_u(\varphi)}_{A_d(\varphi)}, \quad \text{(C.4)}$$

$$J_{\rm int.}\big(A_d;\,A_u\big) = -\int\limits_{A_d(\varphi)}^{A_u(\varphi)}(d\eta)\frac{1}{\sqrt{1-\big(\eta+A_1(\varphi)\big)^2}}\Big\{1+\frac{A_1(\varphi)}{\eta}\Big\}. \quad \text{(C.5)}$$

Having calculated the η-integration, the first term of (C.5) gives arcsin-function, while the second term results in log-function. It reads

$$J_{\rm int.}\big(A_d;\,A_u\big) = -\,\arcsin\big(\eta+A_1(\varphi)\big)\Big|^{A_u(\varphi)}_{A_d(\varphi)} + J^{\rm log.}_{\rm int.}\big(A_d;\,A_u\big) \quad \text{(C.6)}$$

with

$$J^{\rm log.}_{\rm int.}\big(A_d;\,A_u\big) = \frac{A_1(\varphi)}{\sqrt{1-A_1^2(\varphi)}} \times \log\frac{2\big(1-A^2_1(\varphi)\big) - 2\eta A_1(\varphi) + 2\sqrt{\big(1-A_1^2(\varphi)\big)\big(1-(\eta+A_1(\varphi)\big)^2}}{\eta}\Big|^{A_u(\varphi)}_{A_d(\varphi)}. \quad \text{(C.7)}$$

Appendix D. The standard contributions with the linear τ-dependence of DRT

The standard contribution with the linear τ-dependence together with the angular φ-dependence of DRT can be written in the form of a sum as

$$f_S^{(\varphi)}(\vec{\mathbf{x}}) = f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}, \tag{D.1}$$

where

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}} = -\,M \int\limits_0^{\pi/2} (d\varphi)\, \cot\varphi \int\limits_{-A_1(\varphi)}^{A_2(\varphi)} (d\eta) \frac{\eta + A_1(\varphi)}{\eta^2} \tag{D.2}$$

and

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}} = M \int\limits_0^{\pi/2} (d\varphi)\, \tan\varphi \int\limits_{A_2^\star(\varphi)}^{A_4(\varphi)} (d\eta) \frac{\eta + A_1(\varphi)}{\eta^2}. \tag{D.3}$$

For the sake of convenience, we are splitting the contribution $f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}$ into two terms, where one of them resembles the contributions $f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}}$ of (D.2). We have

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}} = f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_1} + f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_2} \tag{D.4}$$

with

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_1} = M \int\limits_0^{\pi/2} (d\varphi)\, \tan\varphi \int\limits_{-A_1(\varphi)}^{A_4(\varphi)} (d\eta) \frac{\eta + A_1(\varphi)}{\eta^2}, \tag{D.5}$$

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_2} = -M \int\limits_0^{\pi/2} (d\varphi)\, \tan\varphi \int\limits_{-A_1(\varphi)}^{A_2^\star(\varphi)} (d\eta) \frac{\eta + A_1(\varphi)}{\eta^2}. \tag{D.6}$$

From (D.6), one can see that

$$A_2^\star(\varphi) \Longrightarrow A_2(\varphi) \quad \text{and} \quad f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}^{N_2} \Longrightarrow f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}} \tag{D.7}$$

provided the replacements:

$$x_1 \leftrightarrow x_2, \quad \sin\varphi \leftrightarrow \cos\varphi. \tag{D.8}$$

At the same time, $A_1(\varphi)$ remains invariant under (D.8).

After the η-integration, we get the following expressions:

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}} = -\,M \int\limits_0^{\pi/2} (d\varphi)\, \cot\varphi \\ \times \Big[\log \frac{x_1\, \cos\varphi - (1-x_2)\, \sin\varphi}{x_1\, \cos\varphi + x_2\, \sin\varphi} + \frac{x_1\, \cos\varphi + x_2\, \sin\varphi}{x_1\, \cos\varphi - (1-x_2)\, \sin\varphi} - 1 \Big] \tag{D.9}$$

and

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_1}_{\text{tan-term}} = M \int\limits_0^{\pi/2} (d\varphi)\, \tan\varphi \times \Big[\log\frac{x_1\,\cos\varphi + x_2\,\sin\varphi - 1}{x_1\,\cos\varphi + x_2\,\sin\varphi} + \frac{x_1\,\cos\varphi + x_2\,\sin\varphi}{x_1\,\cos\varphi + x_2\,\sin\varphi - 1} - 1\Big]. \tag{D.10}$$

As above-mentioned, using the replacements (D.8), the η-integrated contribution of $f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2}_{\text{tan-term}}$ can be derived from $f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}}$. We have

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2}_{\text{tan-term}} = -\, M \int\limits_0^{\pi/2} (d\varphi)\, \tan\varphi \times \Big[\log\frac{x_2\,\sin\varphi - (1-x_1)\,\cos\varphi}{x_1\,\cos\varphi + x_2\,\sin\varphi} + \frac{x_1\,\cos\varphi + x_2\,\sin\varphi}{x_2\,\sin\varphi - (1-x_1)\,\cos\varphi} - 1\Big]. \tag{D.11}$$

For our analysis, it is worth splitting the contributions from (D.9) and (D.11) into the following ones:

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(a)}_{\text{cot-term}} = -\, M \int\limits_0^{\pi/2} (d\varphi)\, \cot\varphi\, \log\frac{x_1\,\cos\varphi - (1-x_2)\,\sin\varphi}{x_1\,\cos\varphi + x_2\,\sin\varphi}, \tag{D.12}$$

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,(a)}_{\text{tan-term}} = -\, M \int\limits_0^{\pi/2} (d\varphi)\, \tan\varphi\, \log\frac{x_2\,\sin\varphi - (1-x_1)\,\cos\varphi}{x_1\,\cos\varphi + x_2\,\sin\varphi}, \tag{D.13}$$

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{(b)}_{\text{cot-term}} = -\, M \int\limits_0^{\pi/2} (d\varphi)\, \cot\varphi\Big[\frac{x_1\,\cos\varphi + x_2\,\sin\varphi}{x_1\,\cos\varphi - (1-x_2)\,\sin\varphi} - 1\Big], \tag{D.14}$$

$$f_S^{(\varphi)}(\vec{\mathbf{x}})\Big|^{N_2,\,(b)}_{\text{tan-term}} = -\, M \int\limits_0^{\pi/2} (d\varphi)\, \tan\varphi\Big[\frac{x_1\,\cos\varphi + x_2\,\sin\varphi}{x_2\,\sin\varphi - (1-x_1)\,\cos\varphi} - 1\Big]. \tag{D.15}$$

Appendix E. The additional contributions with the quadratic τ-dependence of DRT

The additional contributions with the quadratic τ-dependence of DRT are

$$f_A^{(\tau)}(\vec{\mathbf{x}})\Big|^{(a)} = (i\pi)\, M \int\limits_0^{\pi/2} (d\varphi) \frac{A_1(\varphi)}{\sqrt{1-A_1^2(\varphi)}} \theta\big(A_4(\varphi)\big) \times \Big\{\theta\big(-A_2(\varphi)\big) + \theta\big(-A_1(\varphi)\big) + \theta\big(-A_2^\star(\varphi)\big)\Big\} \tag{E.1}$$

and

$$f_A^{(\tau)}(\vec{\mathbf{x}})\Big|^{(b)} = (-i\pi)\, M \int\limits_0^{\pi/2} (d\varphi) \sqrt{1-A_1^2(\varphi)}\, \theta\big(A_4(\varphi)\big) \Big\{\delta\big(A_2(\varphi)\big) + \delta\big(A_2^\star(\varphi)\big)\Big\}. \tag{E.2}$$

Appendix F. The additional contributions with the linear τ-dependence of DRT

The additional contributions with the linear τ-dependence together with the angular φ-dependence of DRT are

$$f_A^{(\varphi)}(\vec{\mathbf{x}}) = f_A^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}} + f_A^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}}, \tag{F.1}$$

where

$$f_A^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{cot-term}} = (-i\pi)\, M \int_0^{\pi/2} (d\varphi)\, \cot\varphi\, \theta\big(A_1(\varphi)\big) \times \Big\{\theta\big(A_2(\varphi)\big) - A_1(\varphi)\delta\big(A_2(\varphi)\big)\Big\} \tag{F.2}$$

and

$$f_A^{(\varphi)}(\vec{\mathbf{x}})\Big|_{\text{tan-term}} = (i\pi)\, M \int_0^{\pi/2} (d\varphi)\, \tan\varphi\, \theta\big(A_4(\varphi)\big) \times \Big\{\theta\big(-A_2^\star(\varphi)\big) - A_1(\varphi)\delta\big(A_2^\star(\varphi)\big)\Big\}. \tag{F.3}$$

We remind that $A_1(\varphi)$, $A_2(\varphi)$, $A_2^\star(\varphi)$, $A_4(\varphi)$ have been defined by (C.2).

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