NATURAL SCIENCE REVIEW 3 100801 (2026)
nsr.jinr.int

Perturbative QCD fitting of KEDR and BESIII e^+e^- data for $R(s)$ and α_s determination

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Abstract

The experimental data collected by the KEDR and BESIII collaborations at the energies below charm quark thresholds are compared with the massless QCD expressions for the e^+e^- annihilation R ratio truncated at different orders of perturbation theory. The fits demonstrate the dependence of the extracted $\alpha_s(M_Z)$ values on the orders of truncation of the corresponding approximations. The next-to-leading order, next-to-next-to-leading order and next-to-next-to-next-to-leading order fits of the combined KEDR data and BESIII data, truncated at the scale of mass of J/Ψ meson, give the following results: $\alpha_s(M_Z) = 0.1151^{+0.0052}_{-0.0069}$, $\alpha_s(M_Z) = 0.1190^{+0.0064}_{-0.0081}$ and $\alpha_s(M_Z) = 0.1283^{+0.0028}_{-0.0075}$. The increasing tendency of fitted $\alpha_s(M_Z)$ value is associated with the effects of kinematical π^2 contributions, not totally controlled within asymptotic perturbation theory expansions, to R-ratio coefficients due to analytical continuation from the space-like to time-like energy regions. The applications of the fixed orders of perturbation theory expansions and careful treatment of the analytical continuation effects are commented.

Keywords: QCD perturbation theory

DOI: 10.54546/NaturalSciRev.100801

1. Introduction

The e^+e^- annihilation into hadrons remains among the most informative processes of high energy physics. Its studies were started in the 1960s [1] and are continued at present at the experimental facilities of the Novosibirsk, Beijing and Tsukuba research centers, even after extensive investigations at CERN, SLAC, and DESY, and by other later e^+e^- colliding machines. The basic theoretical characteristic related to the total cross-section of this process is

$$R(s) = \frac{\sigma_{\rm tot}(e^+e^- \to \gamma^* \to {\rm hadrons})(s)}{\sigma_0(e^+e^- \to \mu^+\mu^-)}, \quad (1)$$

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Received: 10 June 2026; Revised: 22 June 2026; Accepted: 29 June 2026; Published: 21 July 2026; Corrected: 28 July 2026 (changes have been made in Eqs. (12)–(14) and an expression has been added after them)

where $s = (E_{e^+} + E_{e^-})^2$ is the Minkowski squared momentum transfer of the virtual photon γ^*, which is equal to the square of the sum of energies of the incoming real electrons and positrons. In this definition, $\sigma_0(e^+e^- \to \mu^+\mu^-) = 4\pi\alpha^2/(3s)$ is the normalization factor defined by the leading order quantum electrodynamics (QED) massless contribution to the total cross-section of the $e^+e^- \to \gamma^* \to \mu^+\mu^-$ sub-process, and $\alpha = e^2/(4\pi)$ is the tree (namely, without any radiative corrections) expression of the QED coupling constant, which is fixed by the value of the QED fine structure constant. The normalization factor is chosen in this way to take into account only theoretical information on only the interactions between the hadrons, created in the final states, and between their constituent (anti)quarks. While considering $R(s)$ ratio it is accepted by the experimentalists to subtract the QED vacuum polarization corrections to the propagator of real photon γ from the total cross-section of the process $e^+e^- \to \gamma \to$ hadrons and divide it by the radiative QED contributions due to photon radiations from the initial e^+e^- state pairs.

Up to the high energy scale of the manifestation in e^+e^- annihilation of the sub-process of creating a virtual or real Z boson, the most important contributions to the R ratio come from the strong interactions of hadrons between their constituent quarks and anti-quarks via the gluon exchanges. Different aspects related to the determination and theoretical analysis of these important effects were discussed in a number of recent reviews on the subject [2, 3].

In general, the study of R ratio is one of the most fundamental measurements in physics of strong interactions. The measurements performed at different colliders in a wide region of energies starting from the scales of manifestation of light $\rho(770)$, $\omega(782)$ and $\phi(1020)$ mesons and above the thresholds of production of heavier J/Ψ and Υ mesons confirmed that the constituents of observed hadrons, namely, up (u), down (d), strange (s), charmed (c) and beautiful (b) quarks, indeed have fractional electric charges and that the special quantum number of quarks, i.e., color, is equal to three.

In the modern theory of strong interactions, quantum chromodynamics or QCD, it is presumed that, in addition to quarks and anti-quarks, there are also eight massless gluons. Within the method of perturbation theory, their couplings with quarks and anti-quarks allow one to express the prediction for the R ratio as a power series in the energy-dependent QCD coupling constant α_s with definite coefficients. This constant "measures" the force of interactions of the quarks and anti-quarks, bounded inside hadrons, with "gluing" them gluons, and obeys the property of asymptotic freedom, responsible for the inverse logarithmic decrease of this constant with the increase in energy[1].

In order to test the validity of QCD, it is necessary to extract from different processes the value of this coupling constant and compare the results of these extractions between themselves at the definite fixed reference scale. Due to the agreement between theoreticians and experimentalists, this reference scale is fixed to be equal to the mass of the heavy electroweak Z boson M_Z, which was directly observed at the energies of the previously working e^+e^- LEP and SLC colliders, already closed Tevatron machine and of the Large Hadron Collider, which continues to collect data. While fixing this scale as a reference α_s scale, the aim of minimization of theoretical and experimental uncertainties was kept in mind.

In addition to the biennial Particle Data Group Editions (see, e.g., most recent one [4]), there are a number of detailed reviews devoted to the analysis of extractions of α_s from pre-

[1]Note that the effects of the photon radiations from the final quark–anti-quark pairs will be neglected in our considerations.

vious, existing and planned high energy physics experiments [5–8]. However, in these detailed considerations there are no discussions of the status of the extractions of α_s from the R-ratio experimental data. In what follows, we will fill in this gap and will concentrate on certain comments on this topic.

The first theoretically based detailed QCD study was performed in [9], where it was demonstrated that taking into account the positive leading order (LO) α_s correcting effects, determined independently in [10, 11], is really important for demonstration of the realization of the QCD asymptotic freedom property in the analysis of the concrete experimental data obtained at SLAC-based SPEAR e^+e^- collider in the energy region between 2 to 5 GeV and from 5 to 7.8 GeV.

This LO QCD correction was also used in the QCD sum rules analysis of the data of Novosibirsk VEPP-2, Orsay ADONE and Frascati DCI machines in the energy region below 2 GeV, carried out in [12]. It turned out to be important for finding the correlation between the values of α_s and the non-perturbative parameters, describing the effects of condensation of colored gluon and quark fields in the QCD vacuum at rather low energies.

A substantiated study of the problems considered above became possible after the appearance of expressions specified to the theoretical $\overline{\rm MS}$-scheme for the next-to-leading order (NLO) α_s^2-contributions to the R ratio, calculated analytically independently in [13] and almost simultaneously numerically in [14], and confirmed somewhat later by another analytical calculation [15]. More convincing arguments in favor of the validity of QCD as a theory of strong interactions were made in the subsequent work [16], where these positive NLO α_s^2 effects were taken into account in the process of analyzing SLAC SPEAR and DESY PETRA experimental data. Moreover, after measuring the R ratio in the energy region between 14 and 46.8 GeV at the PETRA collider, the members of the CELLO collaboration [17] obtained the $\overline{\rm MS}$-scheme related value for α_s, namely, $\alpha_s(34~\text{GeV}) = 0.169 \pm 0.025$. In this way, the necessity of using information on not huge but positive NLO α_s^2 QCD contributions to the R ratio [13–15] for describing concrete experimental data within the framework of perturbative QCD was clarified.

The results obtained by the CELLO collaboration [17] were further confirmed in the work [18] (with its detailed discussion in [19]) and in the study [20], aimed at more careful α_s-extraction after supplementing the data of the PETRA machine with the existing data from the ADONE, SPEAR, DORIS, CESR, PEP and TRISTAN e^+e^- colliders. Another aim of these works was to include in the studies the theoretical information on the next-to-next-to-leading order (NNLO) α_s^3 QCD effects, obtained from the work [21] with the unfortunate dramatic results. Among them was the finding that the result of [21] for the α_s^3 coefficient used in these studies turned out to be erroneous.

The **corrected negative NNLO** α_s^3 analytical contribution to the R ratio was obtained in [22] after more than a two-year long project. It agreed with the related one, given in [23]. Later on, it was also confirmed by an independent calculation [24]. The re-evaluation project was indeed rather important in view of approaching era of high-energy SLAC SLC and CERN LEP colliders. Almost immediately after publication, the results of [22, 23] were implemented in [25] to analyze $e^+e^- \to$ hadrons data from PETRA, PEP, TRISTAN and other colliders (including LEP). As a result, a high value $\alpha_s(M_Z) = 0.148 \pm 0.019$ (exp.) $\pm\, 0.005$ (theor.) was obtained, which was surprising even to the authors of this work[2].

[2]To avoid the discussions of the effects of electro-weak corrections, we will not consider the further determinations of $\alpha_s(M_Z)$ from LEP data.

In the work of the CLEO collaboration from CESR the R measurement at $\sqrt{s} = 10.52$ GeV energy scale was performed [26]. This implies the following NNLO value of the QCD coupling constant: $\alpha_s(M_Z) = 0.13 \pm 0.005$ (stat.) $\pm\, 0.03$ (syst.) [26]. The more detailed analysis of this data performed later taking into account charm quark mass effects gave a more detailed result: $\alpha_s(M_Z) = 0.124^{+0.013}_{-0.014}$ [27].

The further measurements, performed with the help of the CLEOIII detector, allowed one to get 7 more points for R in the energies between 6.964 and 10.538 GeV and the QCD coupling constant NNLO value $\alpha_s(M_Z) = 0.126 \pm 0.005^{+0.015}_{-0.011}$ [28]. However, the authors of [29] claimed that the inclusion of charm quark mass effects and of the proper matching between the effective theories with four and five flavours changes this result to $\alpha_s(M_Z) = 0.110^{+0.010+0.010}_{-0.012-0.011}$ later on [29]. Another interesting, though unpublished, combined NNLO QCD analysis of the e^+e^- data without the CLEOIII data of [28] resulted in a relatively larger value of $\alpha_s(M_Z)$, namely, $\alpha_s(M_Z) = 0.128 \pm 0.032$ [30].

In the process of determination of the world average combined value $\alpha_s(M_Z) = 0.1180 \pm 0.0009$, the Particle Data Group [4] used another e^+e^- R-ratio related NNLO fixed order perturbation theory result $\alpha_s(M_Z) = 0.1158 \pm 0.0022$, obtained in [31] from the QCD finite energy sum rules analysis of the R-ratio low-energy data to 2 GeV. It was also commented there that the data above 2 GeV did not provide much additional constraint on this analysis.

Not long ago, new more detailed experimental information on R-ratio values in this low energy region, 1.8 GeV $\leqslant \sqrt{s} \leqslant$ 3.7 GeV, was provided by the KEDR collaboration from the Novosibirsk VEPP-4M collider [32–34] and by the BESIII collaboration from the Beijing BEPC-II collider [35] in the region of 2.23 to 3.67 GeV. These data were used recently in [36] in the process of taking into account rather sizable negative next-to-next-to-next-to-leading order (N^{3}LO) α_s^4 contributions to the R ratio, analytically obtained in [37] and confirmed in [38].

In the study [36], the problems of taking into account not yet calculated next-to-next-to-next-to-next-to-leading order (N^{4}LO) α_s^5 effects were also considered. First estimates of these terms were obtained in [37] with a theoretical procedure, developed in [39]. In [36] the results of further more detailed Padé related estimates of [40] were applied, which within specified theory uncertainties turned out to be in agreement with the number of [37] and were also recently confirmed in [41] by another approach.

In [36] the α_s value was not extracted but fixed from the world average value $\alpha_s(M_Z) = 0.1180 \pm 0.0009$, given by the Particle Data Group [4]. In our work, we will follow the way back and consider the problem of extraction of α_s from the analysis of available now R-ratio data of the KEDR collaboration [32–34] and BESIII collaboration [35] in the energy region below the thresholds of production of charm quark and anti-quark pairs and **truncating** R-ratio QCD expression **at each subsequent order** of perturbation theory expansion.

While performing this study, we will try to achieve the following aims:

- to get new e^+e^--related $\alpha_s(M_Z)$ values for the fundamental constant of strong interactions from the independent analysis of these recent experimental data for the R ratio;

- to take into account maximum information on the theoretical perturbative QCD approximations both for experimentally measured R ratio and for the energy evolution of strong interactions coupling constant α_s, obeying the QCD asymptotic freedom property;

- to clarify whether the analysis of existing new lower energy experimental e^+e^- annihilation data is sensitive to the **difference between the sign structures and the values of the QCD perturbation theory coefficients**, originally defined in the Euclidean region, and the ones further transformed into the physical Minkowski region, where the experimental data for the R ratio are obtained.

We will also try to get personal impression on the conclusion made in [36] that the QCD fits give a definite preference to the KEDR data in relation to the BESIII ones for the e^+e^- to hadrons total cross-sections and whether it is possible to minimize the observed [36] definite tension between perturbative QCD predictions for R ratio and the BESIII data by applying definite kinematical cuts to these data.

These cuts disfavor the inclusion of the extracted BESIII 8 data points for R above 3.4 GeV in final physical fits. We will see that taking them into account results in the significant decrease in the goodness-of-fit probability of the outcomes of the fits significantly below the commonly accepted 5% level.

Consequences of underestimating theoretical uncertainties of the N^{3}LO QCD extraction of α_s from the KEDR data [42], where these uncertainties were fixed by choosing the way of fixing them, proposed in [43], but constructively criticized in [44–47], will also be commented.

2. Theory related setup

In theory the expression for the R ratio defined in the Minkowski time-like region is determined by the imaginary part of the photon vacuum polarization function transformed to the complex region:

$$R(s) = 12\pi \mathrm{Im}\Pi(-s + i\epsilon), \tag{2}$$

which enters into the expression of the two-point Green function of the related quark currents, defined in the Euclidean region:

$$\Pi_{\mu\nu}(q) = i \int e^{iqx} < 0|T[J_\mu^{\mathrm{em}}(x) J_\nu^{\mathrm{em}}(0)]|0 > d^4x = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(Q^2), \tag{3}$$

where the Euclidean momentum transfer is $Q^2 = -q^2$ and $J_\mu^{\mathrm{em}}(x) = \sum Q_q \overline{\psi}_q(x)\gamma_\mu \psi_q(x)$ is the vector current of quarks with Q_q fractional charges in units of electron charge. Taking the derivative of the photon vacuum polarization function $\Pi(Q^2)$, it is possible to get the following relations between the characteristics of the e^+e^- annihilation to hadrons in the Euclidean and Minkowski regions:

$$D(Q^2) = -12\pi^2 Q^2 \frac{d\Pi(Q^2)}{dQ^2} = Q^2 \int\limits_0^\infty \frac{R(s)}{(s+Q^2)^2} ds, \quad R(s) = \frac{1}{2\pi i} \int\limits_{-s-i\epsilon}^{-s+i\epsilon} dQ^2 \frac{D(Q^2)}{Q^2}. \tag{4}$$

In the case of three flavours of light active quarks and within the perturbative QCD, the quantities introduced above can be expressed as

$$R_{uds}(s) = 3 \sum_{q=u,d,s} Q_q^2 \left[1 + r_1 a_s(s) + \sum_{k\geqslant 2} r_k(f) a_s(s)^k\right], \tag{5}$$

$$D(Q^2) = 3 \sum_{q=u,d,s} Q_q^2 \left[1 + d_1 a_s(Q^2) + \sum_{k\geqslant 2} d_k(f) a_s(Q^2)^k\right], \tag{6}$$

where f is a number of quarks flavours, $a_s(s) = \alpha_s(s)/\pi$ and $a_s(Q^2) = \alpha_s(Q^2)/\pi$ are the expansion parameters with the related running quark–gluon coupling constants, defined in the Minkowski and Euclidean regions, respectively. Since in the light-quark sector $(\sum_{q=u,d,s} Q_q)^2 = \left(\frac{2}{3} - \frac{1}{3} - \frac{1}{3}\right)^2 = 0$, the extra contributions of order $O(a_s^3)$ proportional to this structure are not considered. In the Euclidean region, the energy dependence of the QCD coupling constant is defined by the solution of the following renormalization-group equation:

$$Q^2 \frac{\partial a_s(Q^2)}{\partial Q^2} = \beta(a_s) = -\sum_{n\geqslant 0} \beta_n a_s^{n+2}. \tag{7}$$

In the $\overline{\text{MS}}$-scheme, usually considered as the reference scheme, in the energy region below the production of charm–anti-charm pairs, the related expressions for the QCD coupling constant, expressed through the order-by-order expansion in the inverse logarithmic terms, up to the N^{4}LO we will be interested in, can be written down as

$$a_s^{\text{NLO}}(Q^2) = \frac{1}{\beta_0 L_{Q^2}} - \frac{\beta_1 \ln L_{Q^2}}{\beta_0^3 L_{Q^2}^2}, \tag{8}$$

$$a_s^{\text{NNLO}}(Q^2) = a_s^{\text{NLO}}(Q^2) + \Delta_{\text{NNLO}}, \tag{9}$$

$$a_s^{\text{N}^3\text{LO}}(Q^2) = a_s^{\text{NNLO}}(Q^2) + \Delta_{\text{N}^3\text{LO}}, \tag{10}$$

$$a_s^{\text{N}^4\text{LO}}(Q^2) = a_s^{\text{N}^3\text{LO}}(Q^2) + \Delta_{\text{N}^4\text{LO}}. \tag{11}$$

The related high-order terms in these expressions are defined as

$$\Delta_{\text{NNLO}} = \frac{\beta_1^2(l_{Q^2}^2 - l_{Q^2} - 1) + \beta_0\beta_2}{\beta_0^5 L_{Q^2}^3}, \tag{12}$$

$$\Delta_{\text{N}^3\text{LO}} = \frac{\beta_1^3(-2l_{Q^2}^3 + 5l_{Q^2}^2 + 4l_{Q^2} - 1) - 6\beta_0\beta_2\beta_1 l_{Q^2} + \beta_0\beta_3}{2\beta_0^7 L_{Q^2}^4}, \tag{13}$$

$$\begin{aligned}\Delta_{\text{N}^4\text{LO}} = {} & \frac{18\beta_0\beta_2\beta_1^3(2l_{Q^2}^2 - l_{Q^2} - 1) + \beta_1^4(6l_{Q^2}^4 - 26l_{Q^2}^3 - 9l_{Q^2}^2 + 24l_{Q^2} + 7)}{6\beta_0^9 L_{Q^2}^5} - \\ & - \frac{\beta_0^3\beta_1\beta_3(12l_{Q^2} + 1) - 10\beta_0^3\beta_2^2 - 2\beta_0^4\beta_4}{\beta_0^9 L_{Q^2}^5},\end{aligned} \tag{14}$$

here $L_{Q^2} = \ln\left(Q^2/\Lambda_{\overline{\text{MS}}}^{(f=3)2}\right)$, $l_{Q^2} = \ln L_{Q^2}$, $\Lambda_{\overline{\text{MS}}}^{(f=3)}$ is the corresponding scale parameter of the world with $f = 3$ number of active quarks flavours, subsequently extracted at each order of perturbation theory. The numerical expressions for the considered coefficients of the QCD β-function read

$$\begin{aligned}&\beta_0 = 2.75 - 0.16667f, \quad \beta_1 = 6.375 - 0.79167f, \\ &\beta_2 = 22.320 - 4.3689f + 0.094039f^2, \\ &\beta_3 = 114.23 - 27.134f + 1.5824f^2 + 0.0058567f^3, \\ &\beta_4 = 524.558 - 181.80f + 17.156f^2 - 0.22586f^3 - 0.0017993f^4.\end{aligned} \tag{15}$$

In the analytical form, the first renormalization-scheme independent LO coefficient β_0 was obtained at the same time in the works [48, 49]. Its discovered positive value served as a cornerstone for understanding the fact that QCD obeys the property of asymptotic freedom. Within gauge-invariant schemes, the second NLO coefficient β_1 is also scheme-independent and was also simultaneously calculated in two works [50, 51]. It was confirmed later on in [52] during analytical calculations of the generally scheme-dependent β_2 coefficient completed in [53]. The result of [53] was confirmed later on in [54]. It is important for getting self-consistent information on the energy dependence of the NNLO approximation of α_s. The next N^{3}LO term was analytically evaluated in [55] and confirmed seven years later in [56]. The fifth β_4-coefficient was calculated in [57] and confirmed in [58, 59]. In our e^+e^- to hadrons related consideration, it will become important in the process of taking into account the estimated N^{3}LO α_s^5-contribution to the R ratio.

We will consider the QCD coupling constant evolution in the time-like domain after the redefinition of $\alpha_s(Q^2)$ to the Minkowski time-like region and redefining perturbation theory expansion in powers of $\alpha_s(s)$ to the expansion in powers of $1/L_s = 1/\ln\left(s/\Lambda^{(f=3)2}_{\overline{\rm MS}}\right)$ function. This redefinition will be made by changing $1/L_Q$ to $1/L_s$ in Eqs. (8)–(11). In general, this shift leads to the appearance of additional π^2-terms in the time-like domain. They arise from the theoretical expression of the time-like version of the QCD coupling constant as the result of using the following analytical continuation procedure: $1/L_Q^{2n} \to 1/(L_s \pm i\pi)^{2n}$. These effects were considered previously in [60–62] and were analyzed within the context of QED in the most recent work [63].

In our further studies we will define the QCD expansion parameter in the Minkowski region as it was defined by Eqs. (8)–(14) in the Euclidean region and trace extra contributions proportional to powers of π^2 analytical continuation in the coefficients for the R ratio

$$r_k(f) = d_k(f) - \Delta_k(f). \tag{16}$$

These analytical continuation effects have the following form:

$$\begin{aligned}
\Delta_1 = \Delta_2 = 0, \quad \Delta_3(f) &= \frac{\pi^2\beta_0^2}{3}d_1, \\
\Delta_4(f) &= \pi^2\left[\beta_0^2 d_2(f) + \frac{5}{6}\beta_0\beta_1 d_1\right], \\
\Delta_5(f) &= \pi^2\left[2\beta_0^2 d_3(f) + \frac{7}{3}\beta_0\beta_1 d_2(f) + \frac{1}{2}\beta_1^2 d_1 + \beta_0\beta_2 d_1\right] - \frac{\pi^4}{5}\beta_0^4 d_1.
\end{aligned} \tag{17}$$

As was shown in an elegant way in [64], in general these analytical continuation effects arise after taking the following integral even powers of the the renormalization-group controllable logarithmic terms:

$$\int\limits_0^\infty dx \frac{\ln^{2n}(x)}{(x+1)^2} = 2(1-2^{1-2n})\zeta_{2n}(2n)!\ n \geqslant 1, \tag{18}$$

where $\zeta_{2n} = \sum\limits_{k\geqslant 1}^{\infty}(1/k)^{2n}$ is the Riemann zeta function with its even values fixed as $\zeta_2 = \pi^2/6$, $\zeta_4 = \pi^4/90$, etc.

Their influence on the change of the asymptotic structure of the perturbative expansion series for the Euclidean Adler D function of Eq. (4) after analytical continuation to the Minkowski region was considered in [39] and later in [65–67]. From a mathematical point of view, the effects of these terms were studied in [68]. They can be summed up using either the contour-improved perturbation theory (CIPT) [69, 70] or its theoretically equivalent analytical perturbation theory (APT) developed in [71, 72].

In what follows, we will use the language of the fixed order perturbation theory, which presumes the existence of the unique expansion parameter α_s in the following $\overline{\rm MS}$-scheme perturbation theory series in the Euclidean and Minkowski regions:

$$D(Q^2) = 3\sum_{q=u,d,s} Q_q^2\big[1 + a_s + 1.6398a_s^2 + 6.3710a_s^3 + 49.076a_s^4 + \sim 275a_s^5 + O(a_s^6)\big], \qquad (19)$$

$$R(s) = 3\sum_{q=u,d,s} Q_q^2\big[1 + a_s + 1.6398a_s^2 - 10.2839a_s^3 - 106.8798a_s^4 - \sim 505a_s^5 + O(a_s^6)\big]. \qquad (20)$$

Notice the change of signs in the corresponding coefficients starting from the $O(a_s^3)$ level up to at least $O(a_s^5)$ terms after taking into account the effects of analytical continuation. The corresponding truncated massless expressions will be used in the process of fits of the concrete experimental data with the aim of defining first the values of the parameter $\Lambda^{(f=3)}_{\overline{\rm MS}}$ and then the related $\alpha_s(m_\tau)$ values.

To get the related $\alpha_s(M_Z)$ values, we first determine the values of $\Lambda^{(f=5)}_{\overline{\rm MS}}$ by transforming the results for $\Lambda^{(f=3)}_{\overline{\rm MS}}$, obtained from the fits, to the world with $f = 5$ numbers of active flavours. This will be carried out by passing the thresholds of production in the e^+e^- annihilation into hadrons process of thresholds of the production of pairs of c and b quarks, fixed by the masses of the related mesons $M_{D^0} = 1.86$ GeV and $M_B = 5.28$ GeV. This will be done successively in every order of perturbation theory, based on the results of the analysis carried out in [73], in accordance with the formulas given below:

$$\beta_0^{(f+1)}\ln\frac{\Lambda^{(f+1)2}_{\overline{\rm MS}}}{\Lambda^{(f)2}_{\overline{\rm MS}}} = \beta_0^{(f)}L_h\left[\left(\frac{\beta_0^{(f+1)}}{\beta_0^{(f)}} - 1\right) + \delta_{\rm NLO} + \delta_{\rm N^2LO} + \delta_{\rm N^3LO} + \delta_{\rm N^4LO}\right], \qquad (21)$$

where $L_h = \ln(M_h^2/\Lambda^{(f)2}_{\overline{\rm MS}})$ depends on the matching scales $M_h = M_{D_0}$ and $M_h = M_B$ when $f = 3$ or $f = 4$ and the high-order perturbative QCD contributions are fixed as

$$\delta_{\rm NLO} = \left(\frac{1}{\beta_0^{(f)}L_h}\right)\left[\left(\frac{\beta_1^{(f+1)}}{\beta_0^{(f+1)}} - \frac{\beta_1^{(f)}}{\beta_0^{(f)}}\right)\ln L_h - \frac{\beta_1^{(f+1)}}{\beta_0^{(f+1)}}\ln\frac{\beta_0^{(f+1)}}{\beta_0^{(f)}}\right], \qquad (22)$$

$$\delta_{\rm NNLO} = \left(\frac{1}{\beta_0^{(f)}L_h}\right)^2\left[\frac{\beta_1^{(f)}}{\beta_0^{(f)}}\left(\frac{\beta_1^{(f+1)}}{\beta_0^{(f+1)}} - \frac{\beta_1^{(f)}}{\beta_0^{(f)}}\right)\ln L_h + \left(\frac{\beta_1^{(f+1)}}{\beta_0^{(f+1)}}\right)^2 - \left(\frac{\beta_1^{(f)}}{\beta_0^{(f)}}\right)^2 - \frac{\beta_2^{(f+1)}}{\beta_0^{(f+1)}} + \frac{\beta_2^{(f)}}{\beta_0^{(f)}} - C_2\right], \qquad (23)$$

$$\begin{aligned}\delta_{\rm N^3LO} = \left(\frac{1}{\beta_0^{(f)}L_h}\right)^3\Bigg[&-\frac{1}{2}\left(\frac{\beta_1^{(f)}}{\beta_0^{(f)}}\right)^2\left(\frac{\beta_1^{(f+1)}}{\beta_0^{(f+1)}} - \frac{\beta_1^{(f)}}{\beta_0^{(f)}}\right)\ln^2 L_h + \\ &+\frac{\beta_1^{(f)}}{\beta_0^{(f)}}\left[-\frac{\beta_1^{(f+1)}}{\beta_0^{(f+1)}}\left(\frac{\beta_1^{(f+1)}}{\beta_0^{(f+1)}} - \frac{\beta_1^{(f)}}{\beta_0^{(f)}}\right) + \frac{\beta_2^{(f+1)}}{\beta_0^{(f+1)}} - \frac{\beta_2^{(f)}}{\beta_0^{(f)}} + C_2\right]\ln L_h + \\ &+\frac{1}{2}\left(-\left(\frac{\beta_1^{(f+1)}}{\beta_0^{(f+1)}}\right)^3 - \left(\frac{\beta_1^{(f)}}{\beta_0^{(f)}}\right)^3 - \frac{\beta_3^{(f+1)}}{\beta_0^{(f+1)}} + \frac{\beta_3^{(f)}}{\beta_0^{(f)}}\right) + \\ &+\frac{\beta_1^{(f+1)}}{\beta_0^{(f+1)}}\left(\left(\frac{\beta_1^{(f)}}{\beta_0^{(f)}}\right)^2 + \frac{\beta_2^{(f+1)}}{\beta_0^{(f+1)}} - \frac{\beta_2^{(f)}}{\beta_0^{(f)}} + C_2\right) - C_3\Bigg]. \end{aligned} \qquad (24)$$

The expressions for $\delta_{\rm NLO}$, $\delta_{\rm NNLO}$ and $\delta_{\rm N^3LO}$ follow from the related calculations of [74–77]. At N^{4}LO, the corresponding correction was also explicitly evaluated in [78]: $\delta_{\rm N^4LO} = \delta\left(\left\{\beta_i^{(f)},\beta_i^{(f+1)}\right\},C_2,C_3,C_4\right)/\left(\beta_0^{(f)}L_h\right)^4$ with $\left\{\beta_i^{(f)},\beta_i^{(f+1)}\right\}=\left(\beta_0^{(f)},\beta_0^{(f+1)},\ldots,\beta_i^{(f)},\beta_i^{(f+1)}\right)$, and $1\leqslant i\leqslant 4$ and C_2,C_3,C_4 are the concrete coefficients evaluated in [75, 76] and [78], respectively. The application of this expression in concrete considerations did not give serious N^{3}LO changes in the $\Lambda^{(f=5)}_{\overline{\rm MS}}$ values. In view of this and to save the space, we will not present the result of [78] in our work explicitly.

It should be noted that due to minor differences between the realizations of the perturbation theory expansions in terms of powers of our $1/L_Q$ or $1/L_s$ expansions and more often considered α_s expansions and of the realizations of the related matching conditions, there are some differences between the final results of application of the related matching procedures based on the formulas mentioned above and those obtained within the framework of the RunDec package of [79, 80]. However, these differences do not exceed 2×10^{-4} in $\alpha_s(M_Z)$ values and we will neglect them in our further considerations.

3. Experimentally related setup

Let us consider first the KEDR collaboration data for the R ratio in the region from 1.84 to 3.72 GeV, given in [32–34]. In our work we use the R_{uds} values published in [34], which contains measurement results for 22 energy points, including the corresponding statistical and systematic uncertainties, as well as correlation matrix for systematic uncertainties. This information allows one to determine the covariance matrix for the KEDR data as

$$C_{ij}^{\rm KEDR}=\delta_{ij}\sigma^2_{{\rm stat},i}+\sigma_{{\rm syst},i}\rho_{ij}\sigma_{{\rm syst},j}, \tag{25}$$

where δ_{ij} is the Kronecker delta. Here $\sigma_{{\rm stat},i}$ and $\sigma_{{\rm syst},j}$ are statistical and systematic errors, and ρ_{ij} is the corresponding correlation matrix, explicitly given in [34].

Let us now define the corresponding χ^2_0 minimization function as

$$\chi^2_0=\sum_i\sum_j\left(R^{\rm exp}(s_i)-R^{\rm th}(s_i)\right)(C^{-1})_{ij}\left(R^{\rm exp}(s_j)-R^{\rm th}(s_j)\right), \tag{26}$$

where $R^{\rm exp}(s_i)$ and $R^{\rm th}(s_i)$ are the measured and theoretically defined values of the R ratio, fixed at the concrete energy points, and C^{-1}_{ij} are the coefficients of the inverse covariance matrix.

In the publication of the BESIII collaboration [35], their data on 14 experimental points for the R ratio in the related energy region were given, but the information on the corresponding correlation matrix was not explicitly specified. In our analysis, we construct this correlation matrix $C_{ij}^{\rm BESIII}$ using the experimental data, given in Table 1 of this work.

Taking into account the comments on systematic uncertainties of BESIII data points for R, given in the related Table 1 caption, namely, the information that "uncertainties from the last four sources" (from luminosity, trigger efficiency, signal model and initial state radiation) "are correlated between the energy points", we estimate the corresponding correlated systematic uncertainty $\sigma_{\rm syst.corr.}$ for every presented BESIII R ratio point, summing these four errors in quadrature. The remaining three uncertainties for event selection, QED background and beam background are used to fix the uncorrelated systematic error $\sigma_{\rm syst.uncorr.}$. The results of our estimates are given in Table 1.

Table 1. BESIII data points used in the fit with total statistical and systematic uncertainties. The second and third columns contain our estimated uncorrelated and correlated systematic uncertainties.

$\sqrt{s}$, GeV	$R(s)$	$\sigma_{\text{syst.uncorr.}}$	$\sigma_{\text{syst.corr.}}$
2.2324	$2.286 \pm 0.008 \pm 0.037$	0.0125	0.0349
2.4000	$2.260 \pm 0.008 \pm 0.042$	0.0143	0.0397
2.8000	$2.233 \pm 0.008 \pm 0.055$	0.0163	0.0531
3.0500	$2.252 \pm 0.004 \pm 0.052$	0.0181	0.0492
3.0600	$2.255 \pm 0.004 \pm 0.054$	0.0189	0.0506
3.0800	$2.277 \pm 0.003 \pm 0.046$	0.0179	0.0424
3.4000	$2.330 \pm 0.014 \pm 0.058$	0.0174	0.0554
3.5000	$2.327 \pm 0.010 \pm 0.062$	0.0217	0.0579
3.5424	$2.319 \pm 0.006 \pm 0.060$	0.0165	0.0575
3.5538	$2.342 \pm 0.008 \pm 0.064$	0.0194	0.0613
3.5611	$2.338 \pm 0.010 \pm 0.066$	0.0206	0.0623
3.6002	$2.339 \pm 0.006 \pm 0.065$	0.0194	0.0618
3.6500	$2.352 \pm 0.009 \pm 0.067$	0.0221	0.0628
3.6710	$2.405 \pm 0.010 \pm 0.067$	0.0234	0.0623

Taking these considerations into account, we now construct the corresponding covariance matrix of BESIII data:

$$C_{ij}^{\text{BESIII}} = \delta_{ij}\left(\sigma^2_{\text{stat},i} + \sigma^2_{\text{syst.uncorr.},i}\right) + \sigma_{\text{syst.corr.},i}\sigma_{\text{syst.corr.},j}. \tag{27}$$

In general, both BESIII and KEDR or any other experimental data may deviate from the expected theoretical expressions. In R measurements, this may be due to the systematic errors say in the luminosity measurements or efficiency determinations. We will take these possible deviations into account in the following way:

- for all experimental points, the minimum correlated systematic uncertainty σ_0 is determined;
- this value is quadratically subtracted from the systematic uncertainties for each point under consideration;
- a new covariance matrix $\tilde{C}_{ij}$ is determined;
- the χ^2 minimization function is redefined as

$$\chi_1^2 = \frac{(\nu-1)^2}{\nu^2\sigma_0^2} + \frac{1}{\nu^2}\sum_i\sum_j \left(\nu R^{\text{exp}}(s_i) - R^{\text{th}}(s_i)\right)(\tilde{C}^{-1})_{ij}\left(\nu R^{\text{exp}}(s_j) - R^{\text{th}}(s_j)\right), \tag{28}$$

where ν is a free normalization parameter, which reflects, to a certain extent, how much, on average, the experimental data differ from the theoretical dependence over the entire energy range under consideration. A minimization function based on only the χ_0^2 function can lead to biased results, since the theoretical curve cannot move over the entire range of measured experimental values of R. Using χ_1^2 allows us to separate the overall bias in the R data from the shape of the theoretical curve, which is actually compared in the fit with its theoretical energy dependence.

This χ_1^2 redefinition procedure was applied previously [18, 19] in the process of NLO fits of R-ratio data above the thresholds of production of c and b quarks. In our work, we use both the χ_0^2 and χ_1^2 minimization functions in the process of fitting e^+e^- hadrons data below the thresholds of production of charm quarks. We consider the approaches based on using minimization functions that take into account possible overall bias as rather useful for estimating extra theory related experimental uncertainties. Note that the introduced parameter ν of the function χ_1^2 is directly related to the systematic uncertainties in more careful determination of the possible uncertainties related to those ones due to variations of efficiency and luminosity.

4. Results of fits of KEDR data

Let us first apply the considerations of the previous section for analyzing KEDR $R(s)$ experimental data given in [32–34]. In the process of concrete fits, made with the help of the standard Root MINUIT library, the perturbative QCD expression truncated at the NLO, NNLO, N^{3}LO and N^{4}LO for the R ratio (see Eq. (20)) and the analogously truncated inverse logarithmic expressions for the QCD coupling constant α_s were used. We do not present errors for results obtained in α_s^5 order since the numerical value for a given contribution to R is only an estimate.

The freedom in choosing the model of Eq. (28) for defining the minimization function χ_1^2 with the additional free parameter ν of the fits reflects the possibility of taking into account additional systematic uncertainties in comparison with the massless perturbative QCD approximation of $R(s)$ with the related 22 KEDR data points fixed from the results of [34] with their minimal experimental systematic uncertainty $\sigma_0 = 0.0171$.

Considering carefully the results given in Table 2, we arrive at the following conclusions:

- the values of the characteristic functions χ_0^2 and χ_1^2 decrease after taking into account explicitly evaluated higher order QCD corrections to the R ratio up to the level of the $O(\alpha_s^4)$ contributions and increase after using the $O(\alpha_s^5)$ estimates;
- the extracted NLO values of $\Lambda_{\overline{\rm MS}}^{(f=3)}$, $\Lambda_{\overline{\rm MS}}^{(f=5)}$ and the related ones for $\alpha_s(m_\tau)$ and $\alpha_s(M_Z)$ increase after taking into account the NNLO and N^{3}LO QCD contributions to the R ratio in the fits;
- the difference between the NLO χ_0^2 related results of Table 2 and the ones of [34] $\Lambda_{\overline{\rm MS}}^{(f=3)} = 361^{+155}_{-174}$ MeV and $\alpha_s(m_\tau) = 0.332^{+0.100}_{-0.092}$ is in part due to the fact that instead of the NLO inverse log α_s approximation of Eq. (8) the NNLO of Eq. (12) was used;
- the introduction of the extra fitting parameter ν into the definition of the minimized characteristic function χ_1^2 leads to a definite decrease in the extracted QCD coupling constant values $\alpha_s(m_\tau)$ and $\alpha_s(M_Z)$;
- the estimates of systematic uncertainties of the fits fixed by the difference of the results obtained in the cases of consideration of two minimization function models χ_0^2 and χ_1^2 are smaller than the errors defined by summing in quadrature the statistical and systematic uncertainties of R ratio data obtained by KEDR;
- the χ_1^2 model motivated NLO $\alpha_s(M_Z)$ value is in good agreement with the Particle Data Group world average value of $\alpha_s(M_Z)$;
- the NLO and NNLO χ_1^2 results $\alpha_s(M_Z) = 0.1150^{+0.0075}_{-0.0110}$ and $\alpha_s(M_Z) = 0.1187^{+0.0093}_{-0.0125}$ are in good agreement with other extractions of $\alpha_s(M_Z)$ from R ratio data in higher energy regions (see [27, 30]);
- the N^{3}LO perturbative QCD contributions to the R ratio move the $\alpha_s(M_Z) = 0.1303^{+0.0025}_{-0.0086}$ value towards higher values, not supported by determination from other processes;
- the inclusion of the estimated N^{4}LO contributions in the fits result in a certain decrease of the N^{3}LO outcomes.

Table 2. Results of fitting KEDR data using two different χ^2 minimization functions for the $O(\alpha_s^2)$, $O(\alpha_s^3)$ NNLO, $O(\alpha_s^4)$ N^{3}LO QCD approximations of $R(s)$ and taking into account the estimated $O(\alpha_s^5)$ N^{4}LO correction. $P(\chi^2)$ represents the p-value (goodness-of-fit probability) for a fit.

$R(s)$ approximation	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	$O(\alpha_s^5)$ estimate
χ_0^2 (KEDR)/ndf	5.941/21	5.454/21	3.677/21	4.448/21
$P(\chi^2)$, %	99.947	99.973	99.999	99.995
$\Lambda_{\overline{\rm MS}}^{(f=3)}$, MeV	447^{+104}_{-115}	490^{+134}_{-137}	672^{+59}_{-135}	605^{+67}_{-127}
$\alpha_s(m_\tau)$	$0.3588^{+0.0663}_{-0.0612}$	$0.4034^{+0.0990}_{-0.0790}$	$0.5793^{+0.0825}_{-0.1319}$	0.485
$\alpha_s(M_Z)$ (NLO)	$0.1189^{+0.0049}_{-0.0062}$			
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV	238^{+70}_{-72}			
$\alpha_s(M_Z)$ (NNLO)		$0.1226^{+0.0061}_{-0.0073}$		
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV		267^{+94}_{-88}		
$\alpha_s(M_Z)$ (N^{3}LO)			$0.1305^{+0.0024}_{-0.0059}$	
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV			392^{+44}_{-96}	
$\alpha_s(M_Z)$ (N^{4}LO)				0.1277
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV				345
χ_1^2 (KEDR)/ndf	5.433/20	5.158/20	3.605/20	3.851/20
P(χ^2), %	99.947	99.965	99.998	99.997
$\Lambda_{\overline{\rm MS}}^{(f=3)}$, MeV	372^{+150}_{-166}	413^{+194}_{-194}	668^{+63}_{-193}	597
ν	0.990 ± 0.014	0.991 ± 0.015	0.997 ± 0.011	0.992
$\alpha_s(m_\tau)$	$0.3178^{+0.0872}_{-0.0807}$	$0.3571^{+0.1313}_{-0.1011}$	$0.5748^{+0.0855}_{-0.1721}$	0.478
$\alpha_s(M_Z)$ (NLO)	$0.1150^{+0.0075}_{-0.0110}$			
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV	190^{+98}_{-98}			
$\alpha_s(M_Z)$ (NNLO)		$0.1187^{+0.0093}_{-0.0125}$		
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV		216^{+132}_{-116}		
$\alpha_s(M_Z)$ (N^{3}LO)			$0.1303^{+0.0025}_{-0.0086}$	
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV			390^{+46}_{-134}	
$\alpha_s(M_Z)$ (N^{4}LO)				0.1273
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV				339

In general, our considerations of the KEDR data are in agreement with the results of the fits of [36] and feel theoretical uncertainties better than the ones given in [42].

5. Results of fits of BESIII data

It is known that there is a definite tension between the N^{3}LO QCD approximation for $R(s)$ obtained in [37] and the experimental results of the BESIII collaboration [35]. This problem was already raised in the original experimental work, where it was emphasized that in the **energy range between 3.4 and 3.6 GeV eight BESIII data points for R lie above the corresponding experimental KEDR results** though at still not disturbing 1.9 σ standard deviations level and are higher at the 2.9 σ standard deviations level than the theoretical N^{3}LO QCD predictions for $R(s)$, provided α_s is fixed as the world average value $\alpha_s(M_Z) \approx 0.118$. Moreover, it was recently emphasized in [36] that the tension between the BESIII results and perturbative QCD expressions remains at more than 3σ level after taking into account the remaining 6 BESIII data points below 3.4 to 2.23 GeV.

In this work, we have made a further step in considering the original BESIII data. In Table 1 we summarized the re-presentation of all 14 BESIII data points taking explicitly into account systematic uncertainties, only commented in the original work [35]. The outcomes of the determination of $\Lambda^{(3)}_{\overline{\rm MS}}$ and α_s values are presented in Tables 4 and 5. In the latter case, the normalization systematic uncertainty factor $\sigma_0 = 0.0349$ in the denominator of Eq. (28) is fixed by the minimal entry to the last column of Table 1.

From the results given in Table 3 it is clear that both ways of fitting the whole set of available BESIII data produce unsatisfactory results. Indeed, in the first case, in addition to an uncomfortably large value of the minimization function $\chi^2_0 \approx 50/13$ (?!) (compare with the values presented in Table 2, which vary between $\chi^2_0 \approx 6/21$ and $\chi^2_0 \approx 4/21$, achieved in our analysis of the KEDR data), the unclear variation of the values of $\Lambda^{(3)}_{\overline{\rm MS}}$ from 150 to 600 MeV with returning to over 230 MeV again was detected. Moreover, the attempt to take into account the possible extra systematic uncertainties in the fits of the BESIII data in addition to the problem of the appearance of high value of over $\chi^2_1 \approx 51/12$ (?!) leads us to the manifestation of the problem of a non-physical small value of the parameter $\Lambda^{(3)}_{\overline{\rm MS}} \approx 20$ MeV (?!).

To understand whether it is possible to get from BESIII any physical information on the problems of "measuring" the value of the strong interaction coupling constant α_s, we excluded from the analysis 8 data points above the mass of J/Ψ meson in the energy region between 3.4 and 3.8 GeV. As was already noticed by the members of the BESIII collaboration, these 8 points are responsible for the certain tensions between the data sets of the BESIII and KEDR

Table 3. Results of fitting all 14 BESIII data points.

$R(s)$ approximation	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	$O(\alpha_s^5)$ estimate
χ^2_0 (BESIII)/ndf	53.723/13	53.603/13	53.484/13	53.386/13
$P(\chi^2)$, %	6.76×10^{-5}	7.10×10^{-5}	7.44×10^{-5}	7.74×10^{-5}
$\Lambda^{(f=3)}_{\overline{\rm MS}}$, MeV	184^{+65}_{-63}	607^{+71}_{-66}	573^{+92}_{-78}	229
χ^2_1 (BESIII)/ndf	51.183/12	51.176/12	51.166/12	53.386/12
$P(\chi^2)$, %	8.65×10^{-5}	8.67×10^{-5}	8.71×10^{-5}	7.96×10^{-5}
$\Lambda^{(f=3)}_{\overline{\rm MS}}$, MeV	19^{+102}_{-18}	18^{+104}_{-17}	20^{+118}_{-19}	20
ν	0.96 ± 0.02	0.96 ± 0.02	0.96 ± 0.02	0.96

Table 4. Results of fitting the truncated BESIII data with points below the mass of J/Ψ meson.

$R(s)$ approximation	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	$O(\alpha_s^5)$ estimate
χ_0^2 (BESIII)/ndf (6 points)	8.128/5	7.377/5	12.596/5	17.778/5
$P(\chi^2)$, %	14.932	19.941	2.747	0.324
$\Lambda_{\overline{\rm MS}}^{(f=3)}$, MeV	527^{+59}_{-67}	607^{+84}_{-89}	573^{+67}_{-86}	477
$\alpha_s(m_\tau)$	$0.4087^{+0.0425}_{-0.0418}$	$0.488^{+0.0781}_{-0.0665}$	$0.478^{+0.0654}_{-0.0678}$	0.3938
$\alpha_s(M_Z)$ (NLO)	$0.1227^{+0.0026}_{-0.0031}$			
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV	292^{+41}_{-45}			
$\alpha_s(M_Z)$ (NNLO)		$0.1280^{+0.0035}_{-0.0040}$		
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV		348^{+61}_{-63}		
$\alpha_s(M_Z)$ (N^{3}LO)			$0.1263^{+0.0029}_{-0.0040}$	
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV			322^{+48}_{-60}	
$\alpha_s(M_Z)$ (N^{4}LO)				0.1218
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV				257
χ_1^2 (BESIII)/ndf (6 points)	6.011/4	5.798/4	5.500/4	6.112/4
$P(\chi^2)$, %	19.832	21.148	23.972	19.094
$\Lambda_{\overline{\rm MS}}^{(f=3)}$, MeV	375^{+130}_{-156}	424^{+173}_{-188}	566^{+111}_{-262}	463
ν	0.966 ± 0.023	0.969 ± 0.024	0.965 ± 0.013	0.955
$\alpha_s(m_\tau)$	$0.3194^{+0.0746}_{-0.0762}$	$0.3638^{+0.1163}_{-0.0994}$	$0.4723^{+0.1136}_{-0.1716}$	0.3849
$\alpha_s(M_Z)$ (NLO)	$0.1151^{+0.0065}_{-0.0101}$			
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV	192^{+84}_{-93}			
$\alpha_s(M_Z)$ (NNLO)		$0.1193^{+0.0082}_{-0.0118}$		
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV		224^{+118}_{-114}		
$\alpha_s(M_Z)$ (N^{3}LO)			$0.1260^{+0.0047}_{-0.0139}$	
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV			317^{+79}_{-170}	
$\alpha_s(M_Z)$ (N^{4}LO)				0.1211
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV				247

collaborations [35]. The results of the fits of these truncated BESIII data are presented in Table 4 and demonstrate agreement with the results of the fits of KEDR data presented in Table 2 within the existing specified concrete uncertainties of both data sets and the performed fits.

6. Combined fits of the KEDR and truncated BESIII data

To get now more statistically substantiated results, we perform the combined fits of 22 data points from the KEDR data and 6 data points of the truncated BESIII data. The adjustment of two sets of data is made by introducing two models for χ_1^2 parameter for the KEDR and BESIII data sets with two extra free parameters ν and ν_1, responsible for fixing systematic uncertainties of the KEDR and BESIII data sets. A similar procedure was previously applied in the combined NLO fits of the available e^+e^- data above the production thresholds of charm and bottom quarks in [19]. The results are presented in Table 5.

Table 5. Outcomes of fitting the truncated BESIII and KEDR data.

$R(s)$ approximation	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	$O(\alpha_s^5)$ estimate
$(\chi_1^2(\text{BESIII}) + \chi_1^2(\text{KEDR}))/\text{ndf}$	11.445/25	10.958/25	9.423/25	10.419/25
$P(\chi^2)$, %	99.050	99.315	99.794	99.536
$\Lambda_{\overline{\text{MS}}}^{(f=3)}$, MeV	374^{+101}_{-113}	420^{+131}_{-136}	618^{+68}_{-162}	518
ν (BES)	0.966 ± 0.019	0.969 ± 0.020	0.964 ± 0.013	0.955
ν (KEDR)	0.990 ± 0.012	0.992 ± 0.013	0.996 ± 0.013	0.990
$\alpha_s(m_\tau)$	$0.3187^{+0.0564}_{-0.0556}$	$0.3607^{+0.0836}_{-0.0725}$	$0.5201^{+0.0769}_{-0.1305}$	0.4203
$\alpha_s(M_Z)$ (NLO)	$0.1151^{+0.0052}_{-0.0069}$			
$\Lambda_{\overline{\text{MS}}}^{(f=5)}$, MeV	191^{+65}_{-68}			
$\alpha_s(M_Z)$ (NNLO)		$0.1190^{+0.0064}_{-0.0081}$		
$\Lambda_{\overline{\text{MS}}}^{(f=5)}$, MeV		220^{+88}_{-84}		
$\alpha_s(M_Z)$ (N^{3}LO)			$0.1283^{+0.0028}_{-0.0075}$	
$\Lambda_{\overline{\text{MS}}}^{(f=5)}$, MeV			354^{+49}_{-111}	
$\alpha_s(M_Z)$ (N^{4}LO)				0.1238
$\Lambda_{\overline{\text{MS}}}^{(f=5)}$, MeV				284

The results of the combined fits of 22 points of the KEDR data set and 6 points of the BESIII data set are illustrated by the plots of Figure 1, where all shifted points of the KEDR and BESIII collaborations given in [34] and [35], are represented as well. The shifts are fixed after the proper adjustments by multiplying them by the related values of the factors ν (KEDR) and ν (BESIII) from Table 5 obtained in the process of the NLO fits. The theoretical curves at these plots are drawn after the substitution of the related fitted outcomes for the range of $\Lambda_{\overline{\text{MS}}}^{(3)}$ values into the $O(\alpha_s^2)$ and $O(\alpha_s^3)$ approximations for $R(s)$ from Eq. (20) with the truncated inverse logarithmic approximations for $\alpha_s(s)$, defined at the NLO, NNLO level by Eq. (8) and Eq. (9), respectively.

Considering the curves of Figure 1, we arrive at the following conclusion:

- in the cases of the obtained NLO and NNLO ranges of values $\Lambda_{\overline{\text{MS}}}^{(3)} = 374^{+101}_{-113}$ MeV and $\Lambda_{\overline{\text{MS}}}^{(3)} = 420^{+131}_{-136}$ MeV, which correspond to the results for $\alpha_s(M_Z)$ given in Table 5, re-

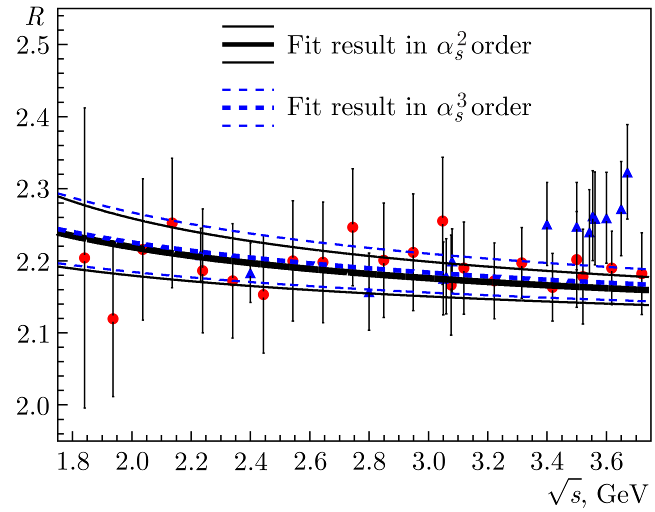

Figure 1. The result of the joint fit of the KEDR (marked with circles) and truncated BESIII (marked with triangles) experimental data. The thin lines correspond to the range specified by Λ parameter errors. The experimental data are presented taking into account the coefficients ν obtained from fitting in α_s^2 order.

spected by the analysis of other experimental data (see the considerations in the reviews of [6–8]) all experimental data points, apart from 6 truncated BESIII points in the energy region above 3.4 GeV, are in agreement with the results of the NLO and NNLO perturbative QCD analysis.

Concerning the deviations of the 6 BESIII data points from the depicted curves, we would like to make the following observations:

- these points lie in the energy region between the masses of the $J/\Psi(3096)$ and $\Psi'(3770)$ mesons, which, due to considerations of the KEDR collaboration, are still well described by the opened production of (anti)quark pairs with $f = 3$ numbers of light flavours only;
- these points belong to the energies of production of lepton $\tau^+\tau^-$ pairs, which can decay further on into light hadrons.

Despite the fact that these physical effects seem to be taken into account in the experimental analysis of both the KEDR and BESIII collaborations discussed above, it may be of interest to further consider the problem of comparison between the points of these data sets in future.

We now consider the energy dependence of different perturbative QCD approximation for $R(s)$ at different orders of expansions in the QCD coupling constant. In Figure 2, we present the related plots for $R(s)$ drawn for the truncated at the NLO, NNLO, NNLO and N^{3}LO terms theoretical expressions for R from Eq. (20) with the QCD coupling constant $\alpha_s(s)$, defined at the NNLO through the inverse log approximation of Eq. (9) with the value of the parameter $\Lambda^{(3)}_{\overline{\rm MS}} = 420$ MeV, obtained in the process of our combined NNLO fits.

Two higher lines correspond to the NLO and NNLO $O(\alpha_s^2)$ and $O(\alpha_s^3)$ approximations for the R ratio. Two lower ones correspond to the not fitted $O(\alpha_s^4)$ and $O(\alpha_s^5)$ related theoretical approximants for $R(s)$.

Notice that in the whole region of energies considered the N^{3}LO perturbative QCD corrections are larger than the NNLO ones. This happens due to the manifestation of the following pattern in general **not asymptotic in the Minkowski region** fixed order perturbative

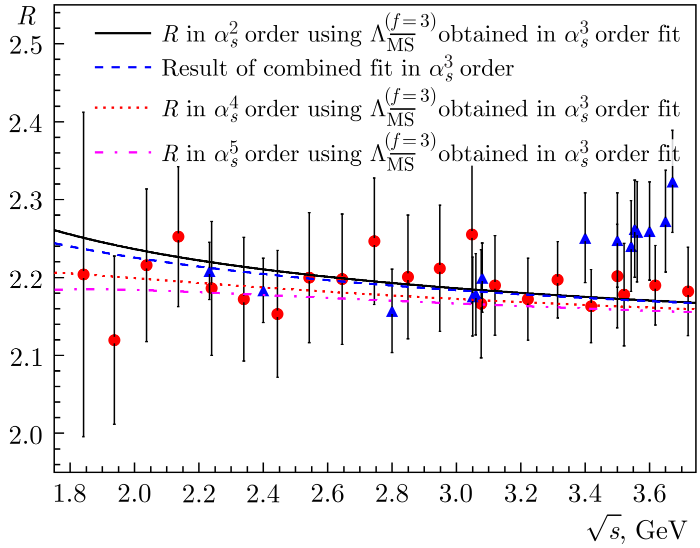

Figure 2. $R(s)$ curves obtained in different orders of perturbative QCD with $\Lambda^{(3)}_{\overline{\rm MS}}$ fixed by the result of joint fit of the KEDR (marked with circles) and truncated BESIII (marked with triangles) data. The experimental data are presented taking into account the coefficients ν obtained from fitting in α_s^2 order.

QCD expression for $R(s)$:

$$R(s) = 2\left[1 + a_s + \underline{1.6398}a_s^2 + \big(\underline{6.3710} - \underline{\underline{16.6550}}\big)a_s^3 + \big(\underline{49.0757} - \underline{\underline{155.9555}}\big)a_s^4 + \right.$$
$$\left. + \big(\underline{\sim 275} - \underline{\underline{780}}\big)a_s^5 + O(a_s^6)\right] . \quad (29)$$

Here the underlined coefficients are the values of the terms directly in the Euclidean region $\overline{\rm MS}$-scheme, which result from the evaluation of the concrete sets of multiloop Feynman diagrams, and the additional terms with double lines are the contributions of the terms shown in Eq. (17), which appear due to the analytical continuation from the Euclidean to the Minkowski region. They are modifying the possibly **asymptotic n! growth** of the related $\overline{\rm MS}$-scheme coefficients of fixed order perturbative QCD expansions in the Euclidean region.

These additional contributions are responsible for the change of the structure of the perturbative QCD expansion from sign constant asymptotic series in the Euclidean region to the non-regular in sign power expansion, which in case of choosing as the value of $\Lambda^{(3)}_{\overline{\rm MS}} = 420$ MeV and thus $\alpha_s(m_\tau)$= 0.361 from the results presented in Table 5 lead to the following numerical behaviour of the perturbative QCD approximation considered in Eq. (29):

$$R(s)|_{s=m_\tau^2} = 2\big[1 + 0.1149 + 0.0217 - 0.0156 - 0.0186 - 0.0101 + \ldots\big]. \quad (30)$$

The appearance of not small negative contributions in the expressions for $R(s)$ is related with the negative contributions to the coefficients of $R(s)$ of sizable kinematical π^2 effects and is responsible for the appearance of the tendency of growth of the extracted values of the QCD coupling constant α_s we observed in the process of applications of the truncated, at different levels, fixed order perturbation theory (FOPT) QCD approximations for $R(s)$ in the energy region with $f = 3$ numbers of active flavours.

To be more definite, in the final part of this work we present the results based on the combined fits of the KEDR and truncated BESIII experimental data:

$$\begin{aligned} \text{NLO} \quad \alpha_s(M_Z) &= 0.1151^{+0.0052}_{-0.0069}, & (31)\\ \text{NNLO} \quad \alpha_s(M_Z) &= 0.1190^{+0.0064}_{-0.0081}, & (32)\\ \text{N}^3\text{LO} \quad \alpha_s(M_Z) &= 0.1283^{+0.0028}_{-0.0075}, & (33)\\ \text{N}^4\text{LO} \quad \alpha_s(M_Z) &= \sim 0.1238, & (34) \end{aligned}$$

which are close to the following similar results of the fits of the KEDR data alone:

$$\begin{aligned} \text{NLO} \quad \alpha_s(M_Z) &= 0.1150^{+0.0075}_{-0.0110}, & (35)\\ \text{NNLO} \quad \alpha_s(M_Z) &= 0.1187^{+0.0093}_{-0.0125}, & (36)\\ \text{N}^3\text{LO} \quad \alpha_s(M_Z) &= 0.1303^{+0.0025}_{-0.0081}, & (37)\\ \text{N}^4\text{LO} \quad \alpha_s(M_Z) &= \sim 0.1273. & (38) \end{aligned}$$

Note that we consider the agreement of our NNLO result with the ones obtained in the process of comparing [42] the KEDR data with the N^{3}LO approximations for R, fixed by applying the procedure proposed in [43] of dealing with high order perturbative QCD expansions, as rather accidental one. Indeed, the application of this procedure to the series for $R(s)$ defined by Eq. (29) presumes further expansion of underlined contributions to the coefficients of perturbative series of Eq. (29) to scale-independent and scale non-independent parts and further absorption of the latter ones, together with double underlined analytical coefficient, into the properly defined scales of α_s in Eq. (29). However, it is known from the considerations of [46, 47] that the analysis of [42, 43] contains the number of still non-fixed ambiguities, which may be fixed in the process of possible further studies. Other interesting issues may be further applications of the low-energy e^+e^- annihilation to hadrons data for the analysis of the number of new theoretical problems, discussed in [81, 82] and quite recenty in [83] in the case of semi-hadronic decay width of τ lepton.

7. Conclusions

We have demonstrated that the direct fits of the KEDR and BESIII collaborations data for the e^+e^- hadrons R ratio below the thresholds of production of charm quarks by the order-by-order truncated perturbation theory QCD expressions for the R ratio and the QCD β function allow one to get new useful information on the values of $\alpha_s(M_Z)$ provided BESIII data are truncated from above by the energy scale of the mass of J/Ψ meson. The new values of $\alpha_s(M_Z)$ that we extracted while applying the NLO and NNLO approximations are in agreement with other determinations of α_s, including the ones from higher energy e^+e^- annihilation to hadrons data. A certain tendency of growth of $\alpha_s(M_Z)$ values starting from NLO to N^{3}LO orders of applied perturbative expansions is detected. Further continuation of the related studies may be useful in clarifying whether it is possible to get more solid information on the value of $\alpha_s(M_Z)$ from further analysis of the e^+e^- annihilation to hadrons $R(s)$ data in the regions below and above the thresholds of production of pairs of charmed quarks for clarifying the possibility of the manifestations of the residual quark-hadron duality-violating effects [84] considered from various points of view in a number of works on the subject [36, 81, 82] and of careful treatment of high order perturbative contributions with the help of CIPT or APT related QCD considerations.

The results of taking into account the explicitly unknown, but estimated, α_s^5 (N^{4}LO) QCD contributions may be further used to simulate in part theoretical uncertainties of the N^{3}LO studies, which within other terms may be more definitely specified by the considerations of the theoretical studies of scale-scheme uncertainties, considered in the detailed reviews [5–7] and studied in various massless analysis of QCD predictions for $R(s)$ in [42] (with the limitations discussed in [46, 47]) and in [85–87] previously. We hope that following these considerations at the α_s^5-level will not change the considered tendency of the non-regular behaviour of the $\overline{\rm MS}$-scheme $O(\alpha_s^4)$ QCD approximations in the region of energies below charm quark–anti-quark production. This feeling may be either supported or restricted by more careful analysis of this problem in future taking into account the dependence on masses of quarks as well.

Acknowledgements

The preliminary results of this work were reported at a scientific seminar at the INR RAS Theory Division on 17 November 2025. We would like to express our gratitude to Konstantin Chetyrkin and in particular to Nikolai Krasnikov for supporting conversations. We also wish to thank Andrej Arbuzov, Dmitry Gorbunov, Ivan Logashenko and Sergey Mikhailov for productive discussions.

The material of the work was further presented at the Quarks-2026 International Seminar, Perozavodsk, 19 May 2026. We would like to thank Diogo Boito, Tatyana Kharlamova and Oleg Zenin for useful comments.

Conflicts of interest

The authors declare no conflicts of interest.

Appendix

The calculations performed in this work were carried out in the approximation of massless quarks. The influence of the s-quark mass can be estimated from the expansions given in [88,

Table 6. Results of fitting the truncated BESIII and KEDR data when considering a non-zero s-quark mass.

$R(s)$ approximation	$O(\alpha_s^2)$	$O(\alpha_s^3)$
$(\chi_1^2(\text{BESIII}) + \chi_1^2(\text{KEDR}))/\text{ndf}$ ($\hat{m}_s = 0.25$ GeV)	11.590/25	11.274/25
$P(\chi^2)$	98.957	99.151
$\Lambda_{\overline{\rm MS}}^{(f=3)}$, MeV	361^{+99}_{-110}	387^{+117}_{-125}
ν (BES)	0.966 ± 0.019	0.967 ± 0.019
ν (KEDR)	0.990 ± 0.012	0.991 ± 0.013
$\alpha_s(m_\tau)$	$0.3124^{+0.0543}_{-0.0539}$	$0.3423^{+0.0700}_{-0.0651}$
$\alpha_s(M_Z)$ (NLO)	$0.1144^{+0.0052}_{-0.0070}$	
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV	183^{+63}_{-66}	
$\alpha_s(M_Z)$ (NNLO)		$0.1172^{+0.0061}_{-0.0078}$
$\Lambda_{\overline{\rm MS}}^{(f=5)}$, MeV		199^{+77}_{-76}

89], which in the QED limit should be compared with the explicit analytical results obtained in [90].

The results obtained are shown in Table 6.

It can be concluded that the non-zero mass of the strange quark (we are using the definition of the invariant strange quark mass $\hat{m}_s$ given, e.g., in [88]) has a noticeable effect on the calculation results, and the corresponding change in the value of $\alpha_s(M_Z)$ is approximately -2×10^{-3}. However, the changes in the results are significantly smaller than the errors in the presented values.

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