

Complexity of Radon transforms

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Abstract

For the reconstruction problem, the universal representation of inverse Radon transforms implies the needed complexity of the direct Radon transforms which leads to additional contributions. In the standard theory of generalized functions, if the outset (origin) function which generates the Radon image is a pure-real function, as a rule, the complexity of Radon transforms becomes in question. In the paper, we discuss the Fourier slice theorem analyzing the degenerated (singular) points as possible sources of the complexity. We also demonstrate different methods to generate the needed complexity at the intermediate stage of calculations. Besides, we show that the introduction of the hybrid (Wigner-like) function ensures naturally the corresponding complexity. The discussed complexity not only provides the additional contribution to the inverse Radon transforms, but also makes an essential impact on the reconstruction and optimization procedures within the framework of the incorrect problems. The presented methods can be effectively used for the practical tasks of reconstruction problems.

Keywords: reconstruction problem, Radon transformations, inversion problem

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1. Introduction

In the modern world, computed tomography technologies (CT technologies) influence many different fields. Basically it is true owing to a possibility to investigate the internal composite structure of an object without cutting and breaking. From the mathematical viewpoint, CT technologies are closely associated with the application of both the direct and inverse Radon transforms [1]. It is well known that the inverse Radon transforms allow us to visualize the internal structure of the objects under investigation. The quality of visualization strongly depends on the inversion procedure of Radon transforms that, as is known, is ill-posed and it demands the corresponding regularization, see, for example, [2].

In [3, 4], the universal (or unified) representation of the inverse Radon transform has been proposed and studied for any dimension of space. Thanks to the use of generalized function (distribution) theory¹, the mentioned universal representation involves the new contribution (in comparison with the standard methods based on the Courant–Hilbert identities [6]) which

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¹By definition, the generalized function (or distribution) is given by the corresponding functional defined via the integration with the finite function, see, for example, [5].

definitely affects the reconstruction. On the other hand, it extends the Tikhonov-like regularization (AC regularization) used for the solutions of the incorrect task classes. In particular, within the reconstruction procedure, the additional new contribution to the unified inverse Radon transforms is given by the integration with the complex measure that should be compensated, generally speaking, by the complexity of direct Radon image.

In the present paper, we analyze in detail the Fourier slice theorem which plays an extremely important role for the derivation of the universal inverse Radon transforms. Moreover, we give the interpretations of degenerated (singular) points which require a certain regularization and, as a consequence, the corresponding complexity.

We also propose different methods on how to generate the needed complexity at the intermediate stage of calculations. Our methods are based on (i) the demonstrated non-trivial holonomy of Radon transforms; (ii) the Fourier series expansion applied within the wide-spread discrete slice approximation [1] together with the use of the hybrid (Wigner-like) functions. The latter are, by definition, the functions of both spatial and momentum coordinates. For the universal inverse Radon transforms and, then, for the reconstruction procedure, this trick has been never used up to now.

2. The Fourier slice theorem and the degenerated (singular) points

In this section, we analyze in detail the Fourier slice theorem which is the principal step toward the universal inverse Radon transforms.

Let $f(\vec{x})$ with $\vec{x} \in \mathbb{R}^2$ (this is x -space) be an original outset function the internal structure of which is under investigation without breaking. For the sake of simplicity, we focus on the two-dimensional coordinate space; the extension up to an arbitrary dimension is obvious, see, for example, [4].

Theorem: *the Fourier image of the outset function $f(\vec{x})$ relates to the direct Radon image of the same function $f(\vec{x})$ through the one-dimensional Fourier transformation with respect to the radial coordinate:*

$$\mathcal{F}[f](\lambda, \varphi) = \int_{-\infty}^{+\infty} (d\tau) e^{-i\lambda\tau} \mathcal{R}[f](\tau, \varphi). \quad (2.1)$$

Here and in what follows, the brackets in the integration measure, see $(d\tau)$, denote that the corresponding normalization factors have been included and they are not written explicitly unless it leads to misunderstanding.

On the l.h.s. of (2.1), the Fourier image of $f(\vec{x})$ is given by

$$\mathcal{F}[f](\vec{q}) = \int_{-\infty}^{+\infty} d^2\vec{x} e^{-i\langle\vec{q}, \vec{x}\rangle} f(\vec{x}) \quad (2.2)$$

and the polar coordinate system: $\vec{q} \equiv \lambda \vec{n}_\varphi$ with $|\vec{n}_\varphi| = 1$ is being implied in q -space that is conjugated to x -space. While, on the r.h.s. of (2.1), the direct Radon transform of $f(\vec{x})$ is defined as

$$\mathcal{R}[f](\tau, \varphi) = \int_{-\infty}^{+\infty} d^2\vec{x} f(\vec{x}) \delta(\tau - \langle\vec{n}_\varphi, \vec{x}\rangle), \quad (2.3)$$

where the argument of delta function parametrizes the straightforward line in $\mathbb{R}^2$. As seen from (2.1), the angular dependence of Fourier and Radon images are coinciding.

To prove the inference of (2.1), we insert the integral unit into the r.h.s. of (2.2). It reads

$$\mathcal{F}[f](\vec{q}) = \int_{-\infty}^{+\infty} d^2\vec{x} e^{-i\langle\vec{q},\vec{x}\rangle} f(\vec{x}) \times \left\{ \int_{-\infty}^{+\infty} (dt) \delta(t - \langle\vec{q},\vec{x}\rangle) \right\}. \quad (2.4)$$

Introducing

$$\xi \stackrel{\text{def.}}{=} \frac{q_2}{q_1}, \quad z \stackrel{\text{def.}}{=} \frac{t}{q_1}, \quad (2.5)$$

one can readily obtain

$$\mathcal{F}[f](q_1, \xi q_1) = \int_{-\infty}^{+\infty} (dz) e^{-izq_1} \left\{ \int_{-\infty}^{+\infty} dx_2 f(z - \xi x_2, x_2) \right\} \equiv \int_{-\infty}^{+\infty} (dz) e^{-izq_1} \mathcal{R}[f](z, \xi), \quad (2.6)$$

which can be represented in the form of (2.1) if the polar system has been used and the variable $\tau = t/\lambda$ has been introduced instead of z , see (2.5).

Notice that the Fourier variable $\lambda \in [0, +\infty]$ (the radial component of q -space) and the Radon radial variable $\tau \in [-\infty, +\infty]$ form the conjugated set. Hence, the singular point $\lambda = 0$ (it is the so-called “axis” singularity of the polar system), leading to the degeneration of angular dependence [4], corresponds to the point $\tau = \pm\infty$. If the Radon image as a function of arguments is bounded and is finite for some reasons, the axis singularity is not important. On the other hand, the point $\lambda = +\infty$ may generate the other type of singularity and, therefore, the conjugated Radon variable $\tau = 0$ should also be regularized [4].

The mentioned situation appears in the inversion problem where both sides of (2.1) have been integrated out over the radial and angular variables. Indeed, having used the inverse Fourier transform together with (2.1), we can write down that

$$f(\vec{x}) = \int_{-\infty}^{+\infty} d^2\vec{q} e^{+i\langle\vec{q},\vec{x}\rangle} \mathcal{F}[f](\vec{q}) \Big|_{\vec{q}=\lambda\vec{n}_\varphi} \quad (2.7)$$

$$= \int_0^{+\infty} d\lambda \lambda \int_{\text{f.r.}} d\varphi e^{+i\lambda\langle\vec{n}_\varphi,\vec{x}\rangle} \mathcal{F}[f](\lambda, \varphi), \quad (2.8)$$

where “f.r.” implies that the angular integration measure covers the full regions of variations. In the inverse Fourier transform, see (2.7), the integration in q -space should be understood in a sense of the principle value integration. It is needed to avoid the problems with the existence of improper integrations (see, for example, [5]).

Hence, the use of the Fourier slice theorem, see (2.1), yields²

$$f_\varepsilon(\vec{x}) = \int_{\text{f.r.}} d\varphi \int_{-\infty}^{+\infty} (d\eta) \mathcal{R}[f](\eta + \langle\vec{n}_\varphi,\vec{x}\rangle, \varphi) \times \left\{ \int_0^{+\infty} d\lambda \lambda e^{-i\lambda\eta} \Big|_{\varepsilon\text{-reg.}} \right\}, \quad (2.9)$$

where, in the factorized λ -integration, “ ε -reg.” denotes the necessary regularization of λ -integration.

² ε as a subscript of f denotes the ε -regularization that should be used in (2.9).

Since, in (2.9), the λ -integration has been factorized out, it can be independently regularized as the regularization of η -pole (which occurs after the integration over λ):

$$\delta_+(\eta) = \int_0^{+\infty} (d\lambda) e^{-i\lambda\eta} \Big|_{\varepsilon\text{-reg.}} \equiv \int_0^{+\infty} (d\lambda) e^{-i\lambda(\eta-i\varepsilon)} = \frac{1}{\eta - i\varepsilon}. \quad (2.10)$$

Alternatively, for regularization, we can make a replacement: $\eta \rightarrow \eta - i\varepsilon$ in the integration over η in (2.9). It results in the complex continuation of the Radon radial component the necessity of which agrees with the dual Radon transforms, see [2–4].

To conclude this section, the imaginary part of $\delta_+(\eta)$ of (2.10) disappears provided the full region of angular integration is considered [2–4]. It is worth stressing that the full regions of angular and radial integrations in the polar system correspond to the full region of integration assumed, from the very beginning, in the Cartesian system of q -space, see (2.7). Moreover, it is already clear that (2.9) establishes a basis for the universal inversion of Radon transforms in the case of the restricted angular integration [4].

3. The linearity and outset functions with defects

The linearity of the direct Radon transforms:

$$\mathcal{R}[f_1 + f_2](\tau, \varphi) = \mathcal{R}[f_1](\tau, \varphi) + \mathcal{R}[f_2](\tau, \varphi), \quad (3.1)$$

is one of the basic properties that should be satisfied. The condition (3.1) is valid provided, in particular, both outset functions have the full-region supports and they have the regular properties. Besides, the corresponding integration measures that define the transformations, as a rule, cover also the full region [1, 5, 7].

Based on the applications of Radon transforms in CT technology, the outset function can be presented as a combination of functions which are differently localized in x -space. In this section, we dwell on this case, dictated by practice. The latter can also lead to the complexity of Radon transforms.

At the beginning, let $f_1(\vec{x})$ and $f_2(\vec{x})$ be $g_1(\vec{x}) \Theta_I(\vec{x})$ and $g_2(\vec{x}) \Theta_{III}(\vec{x})$, respectively, where $\Theta_{I,III}$ are the characteristic indicators defined as

$$\Theta_I(\vec{x}) = \begin{cases} 1, & \text{if } \vec{x} \in \{\Omega_I | x_i \geq 0\} \\ 0, & \text{if } \vec{x} \notin \{\Omega_I | x_i \geq 0\} \end{cases}, \quad \Theta_{III}(\vec{x}) = \begin{cases} 1, & \text{if } \vec{x} \in \{\Omega_{III} | x_i < 0\} \\ 0, & \text{if } \vec{x} \notin \{\Omega_{III} | x_i < 0\} \end{cases}. \quad (3.2)$$

As mentioned, in the direct Radon transform (2.3), the argument of delta function gives the line parametrization as a function of the radial τ and angular φ variables which define the position of the straightforward line in x -space. On the other hand, the direct Radon transform can be reduced to the curve-linear integral of the first kind over the parameter s that determines the integration measure along the given line, see [4].

If the outset function is combined as

$$G(\vec{x}) = g_1(\vec{x}) \Theta_I(\vec{x}) + g_2(\vec{x}) \Theta_{III}(\vec{x}), \quad (3.3)$$

the functional associated with the Radon transform is given by

$$\mathcal{R}[G](\tau, \varphi) \stackrel{\text{def.}}{=} \int_{-\infty}^{+\infty} d^2\vec{x} \{g_1(\vec{x}) \Theta_I(\vec{x}) + g_2(\vec{x}) \Theta_{III}(\vec{x})\} \delta(\tau - \langle \vec{n}_\varphi, \vec{x} \rangle). \quad (3.4)$$

Hence, one can see that the components of $\vec{x}$ should simultaneously satisfy two conditions:
 (a) the delta function singles out the points which lie on the given straightforward line and
 (b) the points have to belong to either Ω_I or Ω_{III} due to the characteristic indicators.

Focusing on the case of $\tau > 0$ and $\varphi \in [0, \pi/2]$, we get the following³:

$$\mathcal{R}[G](\tau, \varphi) \Big|_{\varphi \in [0, \pi/2]}^{\tau > 0} = \int_{-\infty}^{+\infty} d^2 \vec{x} g_1(\vec{x}) \Theta_I(\vec{x}) \delta(\tau - \langle \vec{n}_\varphi, \vec{x} \rangle) = \mathcal{R}[g_1](\tau, \varphi). \quad (3.5)$$

Then, we make a shift: $\varphi \rightarrow \varphi + \pi$ in $\mathcal{R}[G](\tau, \varphi)$ of (3.4), and get

$$\mathcal{R}[G](\tau, \varphi + \pi) \Big|_{\varphi \in [0, \pi/2]}^{\tau > 0} = \int_{-\infty}^{+\infty} d^2 \vec{x} g_2(\vec{x}) \Theta_{III}(\vec{x}) \delta(\tau - \langle \vec{n}_{\varphi+\pi}, \vec{x} \rangle) = \mathcal{R}[g_2](\tau, \varphi + \pi). \quad (3.6)$$

As the next step, we perform another shift by π in $\mathcal{R}[g_2]$ on the r.h.s. of (3.6) to obtain

$$\int_{-\infty}^{+\infty} d^2 \vec{x} g_2(\vec{x}) \Theta_{III}(\vec{x}) \delta(\tau - \langle \vec{n}_{\varphi+\pi+\pi}, \vec{x} \rangle) = \mathcal{R}[g_2](\tau, \varphi + \pi + \pi) = 0 \quad (3.7)$$

owing to the characteristic indicator Θ_{III} . However, on the other hand, the shift by 2π of $\mathcal{R}[G](\tau, \varphi)$, see (3.4), leads to

$$\mathcal{R}[G](\tau, \varphi + 2\pi) \Big|_{\varphi \in [0, \pi/2]}^{\tau > 0} = \mathcal{R}[g_1](\tau, \varphi) \quad (3.8)$$

and, therefore, we have

$$\boxed{\mathcal{R}[g_2](\tau, \varphi + \pi + \pi) \neq \mathcal{R}[g_1](\tau, \varphi).} \quad (3.9)$$

The condition (3.9) points to the existence of the non-trivial holonomy which can generate the complexity discussed in the present paper. In addition, owing to the different properties of g_1 and g_2 of (3.3), it is not excluded that

$$\lim_{\tau \rightarrow 0-} \mathcal{R}[g_2](\tau, \varphi) \neq \lim_{\tau \rightarrow 0+} \mathcal{R}[g_1](\tau, \varphi), \quad (3.10)$$

giving the disconnection which is essential for the contributions of $F_A, \tilde{F}_A$, see below (5.4).

Having used the mentioned demonstration, we are in a position to discuss the outset function which contains some “defect”. We assume that

$$\tilde{G}(\vec{x}) = [g_1(\vec{x}) + g_2(\vec{x})] \Theta_I(\vec{x}) + g_2(\vec{x}) \Theta_{III}(\vec{x}), \quad (3.11)$$

where g_1 plays now a role of some “defect” that is localized in Ω_I .

We again consider the rotation by 2π in the direct Radon transforms by two ways: (i) $\varphi \rightarrow \varphi + 2\pi$ and $\varphi \rightarrow \varphi + \pi + \pi$ (with the calculation of Radon functional at the intermediate point $\varphi + \pi$). Having used the results of (3.5)–(3.9), we readily derive that

$$\boxed{\mathcal{R}[g_2](\tau, \varphi + \pi + \pi) \neq \mathcal{R}[g_1 + g_2](\tau, \varphi).} \quad (3.12)$$

³In a similar manner we can treat the case of $\tau > 0$ and $\varphi \in [\pi, 3\pi/2]$.

This allows us to extract the influence of the “defect” described by g_1 on the common “background” in Ω_I and Ω_{III} given by g_2 .

To summarize this section, we have presented the evidences for the non-trivial holonomy of Radon transforms. In particular, we have suggested the way how to single out the effects from the so-called “defects” hidden into the given functions describing the object under CT investigation.

4. Two-dimensional space versus three-dimensional space

In [4], the universal and unified representation for the inversion of Radon transforms has been presented where the Courant–Hilbert identities [6] have not been applied. We remind that the Courant–Hilbert identities are based on the use of the Green formulae the different forms of which depend on either even or odd dimension of space [5, 7]. For the practical uses (for example, in medicine), the most needed cases correspond to $\mathbb{R}^3$ and $\mathbb{R}^2$ spaces.

However, as dictated by the application of CT technology, different algorithms have been designed for the two-dimensional reconstructions which work with the corresponding transverse projections of the three-dimensional object.

In other words, if the object, which is needed to be reconstructed, is described by the three-dimensional origin outset function $f(x_1, x_2, x_3)$, we perform the finite number of transverse sections, say, regarding x_3 -axis:

$$f(x_1, x_2, x_3) \implies \left\{ f(x_1, x_2, x_{3_1}), f(x_1, x_2, x_{3_2}), \dots, f(x_1, x_2, x_{3_n}) \right\}, \quad (4.1)$$

where the number of sections should be defined by the experiment. The segmentation given by (4.1) is known as *a discrete slice method* known in CT technology. In this case, we always deal with the direct and inverse Radon transforms determined on the two-dimensional space formed by (x_1, x_2) , while the third discrete coordinate plays a role of the external parameter [1].

5. The hybrid Wigner-like function and the complex Radon transforms

As explained in a series of papers [2–4], the regularized inverse Radon operator involves two contributions. If one of them is related to the real integration measure, the other is associated with the imaginary integration measure (that is a result of the Cauchy theorem).

According to the scenario described in [3], the term of the inverse Radon operator with the imaginary measure is extremely important for the optimization procedure which works with the condition inspired by the corresponding norms. On the other hand, in QFT, even the direct Radon transform, which is linked to the corresponding transverse-momentum dependent parton distributions, can possess the imaginary part owing to the interactions in the correlators [2]. In this section, excepting QFT models, we study the natural source of complexity which appears in the Radon transforms.

In this section, we now begin with $\mathbb{R}^3$ -space where the coordinate system has been defined. Let $f(\vec{x})$, with $\vec{x} \in \mathbb{R}^3$, be an outset (origin) function which is usually well-localized. In the reconstruction problem, the form of $f(\vec{x})$ has to be restored owing to the inverse Radon transform.

Since the integral representation of universal inverse Radon transform involves the additional term which should be related to the complex integrand given by the direct Radon transform, we

first introduce the hybrid (Wigner-like) function⁴ associated with the outset function. Namely, we perform the direct Fourier transform regarding one of three coordinates applied to $f(\vec{x})$. We have

$$F(x_1, x_2; k) = \int_{-\infty}^{+\infty} (dx_3) e^{-ikx_3} f(x_1, x_2, x_3), \quad (5.1)$$

where, as used above, the irrelevant normalization has been absorbed into the integral measure. In (5.1), it is clear that $f(x_1, x_2, x_3) \in \Re$ by definition, while $F(x_1, x_2; k) \in \mathbb{C}$ already. Hence, in the discrete slice method, see (4.1), we have the following:

$$\tilde{F}(x_1, x_2; k) = \sum_n e^{-ikx_{3n}} f(x_1, x_2, x_{3n}), \quad \tilde{F}(x_1, x_2; k) \in \mathbb{C}, \quad (5.2)$$

which is nothing but the Fourier series expansion.

Then, we calculate the Radon image of the Fourier F -functions as

$$\mathcal{R}[F; \tilde{F}](\tau, \varphi; k) = \int_{-\infty}^{+\infty} d^2\vec{x} \left[\begin{array}{c} F(x_1, x_2; k) \\ \tilde{F}(x_1, x_2; k) \end{array} \right] \delta(\tau - \langle \vec{n}_\varphi, \vec{x} \rangle), \quad (5.3)$$

where $\mathcal{R}[F; \tilde{F}](\tau, \varphi; k) \in \mathbb{C}$ and the unit vector is given by $\vec{n}_\varphi = (\cos \varphi, \sin \varphi)$. Notice that in (5.3) the integration measure corresponds to the two-dimensional space, while the momentum k plays a role of the external parameter.

Hence, the inverse Radon transform in the universal form is given by

$$\boxed{F_\varepsilon(x_1, x_2; k) = F_S(x_1, x_2; k) + F_A(x_1, x_2; k)}, \quad (5.4)$$

where

$$F_S(x_1, x_2; k) = - \int_{-\infty}^{+\infty} (d\eta) \frac{\mathcal{P}}{\eta^2} \int d\mu(\varphi) \mathcal{R}[F](\eta + \langle \vec{n}_\varphi, \vec{x} \rangle, \varphi; k) \quad (5.5)$$

and

$$F_A(x_1, x_2; k) = -i\pi \int d\mu(\varphi) \frac{\partial}{\partial \eta} \mathcal{R}[F](\eta + \langle \vec{n}_\varphi, \vec{x} \rangle, \varphi; k) \Big|_{\eta=0}. \quad (5.6)$$

Similar expressions can be written for $\tilde{F}$ -function due to the trivial replacement.

In practical applications, the values of $\mathcal{R}[F; \tilde{F}]$ are usually known from the experiment (or from the observation). Restoring the function F or $\tilde{F}$ from $\mathcal{R}[F; \tilde{F}]$, one can then reconstruct the function f . In this connection, based on the slice method (4.1), the introduction of $\tilde{F}$ aggregates all two-dimensional slices with the help of the continuous parameter k , see (5.2), which is a very convenient parameter for the optimization procedure [3].

Therefore, for the reconstruction, the ultimate scheme can be expressed as

$$\boxed{\mathcal{R}[F; \tilde{F}](\tau, \varphi; k) \xrightarrow{\mathcal{R}^{-1}} \{F(x_1, x_2; k), \tilde{F}(x_1, x_2; k)\} \implies f(x_1, x_2, x_3)}. \quad (5.7)$$

This scheme illustrates the principle result which shows us that the necessary step to obtain the complexity of direct Radon transform is given thanks to the introduction and the transition to

⁴In the particular case of QFT, the Wigner function corresponds to the quasi-probability distribution which is actually a real function only.

the hybrid functions $F(x_1, x_2; k)$ or $\tilde{F}(x_1, x_2; k)$. Notice that the use of Fourier images in the forms of the hybrid F -functions, see (5.1) and (5.2), gives one of possibilities to deal with the complex direct Radon transforms. We remind that F_S and $\tilde{F}_S$ are determined by $\Re \{ \mathcal{R}[F; \tilde{F}] \}$ and F_A and $\tilde{F}_A$ are related to $\Im \{ \mathcal{R}[F; \tilde{F}] \}$, guaranteeing the new important contribution within the reconstruction problem.

6. Conclusions

In the paper, we have described the methods where the complexity of the Radon transform appears naturally. In the connection with the reconstruction problem, the use of the complex Radon transforms is very important for the computational scheme. This is because of the additional contribution existence that plays a crucial role in the improved visualization procedure [3].

We have demonstrated the conditions for the non-trivial holonomy of Radon transforms leading to the complexity of transformations. In addition, we have also presented the method to extract the “defect” hidden into the common “background” function. This opens a window for the development of new reconstruction schemes.

Furthermore, we have presented the method which is practically identical to the Fourier series expansion applied within the framework of the discrete two-dimensional slice approximation [1]. In its turn, the discrete two-dimensional slice approximation is a wide-spread approximation in the medical CT technologies. The proposed method is based on the use of the hybrid (Wigner-like) functions depending simultaneously on the spacial and momentum coordinates. The introduced hybrid function as a complex function can be considered as a very convenient tool for the universal inverse Radon transforms. This trick has been first applied in the context of the inversion procedure of Radon transforms.

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Conflict of interest

The author declares no conflict of interest.

Data Availability Statement

This manuscript has no associated data or the data will not be deposited.

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