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A century of the Bose–Einstein condensation concept and half a century of the JINR experiments for observation of condensate in superfluid ^{4}He (He II)

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Abstract

This short review is devoted to the celebration of two major events in quantum physics. The first one is the birth of the concept of Bose–Einstein condensation (1925) and the second is the experimental proof that it does exist and appears in liquid ^{4}He *simultaneously* with superfluidity below the λ-point (1975).

Both of these events are tightly related to the Bogoliubov theory of superfluidity (1947). The existence of condensate in the system of interacting bosons is the key *ansatz* of this theory. Therefore, the experiments started at JINR-Dubna in 1975 confirmed this prediction of the Bogoliubov theory that superfluidity of liquid ^{4}He (He II) should emerge at the same time as the Bose–Einstein condensation.

Keywords: Bose–Einstein statistics and condensation, conventional and non-conventional condensations, generalised condensations, Bogoliubov theory of superfluidity, deep-inelastic neutron scattering, Bose–Einstein condensate in liquid ^{4}He, JINR-Dubna

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Dédié à Iridii Aleksandrovich Kvasnikov, qui m'a appris à admirer la condensation de Bose–Einstein...

1. Bosons and Bose–Einstein Condensation (1925)

"Autant les physiciens sont inventifs dans leur capacité à poser des questions sur la Nature et à l'expliquer, autant les mathématiciens sont tout aussi inventifs pour expliquer pourquoi les physiciens font les choses correctement..."

"Ich behaupte aber, dass in jeder besonderen Naturlehre nur so viel eigentliche Wissenschaft angetroffen werden könne, als darin Mathematik anzutreffen ist."
Immanuel Kant, *Metaphysische Grundlagen der Naturwissenschaft, 1786.*

1.1. Many-body quantum systems and the Bose statistics

1.1.1. Planck formula

The *condensation* of the Bose gas predicted by A. Einstein [1] one hundred years ago (1925) went through more than a decade of a strong doubt before it became widely accepted after a convincing elucidation and formal mathematical arguments by F. London [2] in 1938.

Although at that time even the Quantum Mechanics of few particles had not yet been completely formulated (the Schrödinger equation was published only in 1926 [3]), physicists were already puzzled by the problem of describing the conductivity, or specific heat, of *many-body* quantum systems such as electrons in metals. Another problem of this kind concerned a formula for the spectral density of electromagnetic radiation emitted by a black-body in thermal equilibrium at given temperature T. Once M. Planck [4] had discovered for it an empirically fitting law, the *Planck formula* (1900), it was necessary to find satisfactory (mathematical) arguments which yield a derivation of this formula.

Planck's arguments were formal and they were based on two hypotheses:

— The first one was to attribute to any *infinitesimal* band $d\nu$ (for *mode* ν) of radiation in a closed cavity at given temperature T a system (proportional to *volume* of the cavity) of N monochromatic vibrating *resonators* with the proper frequency ν.

— The second hypothesis was that the energy ε_n of each resonator is *quantified* according to the law: $\varepsilon_n = n\,h\nu$. Here $n = 0, 1, 2, \ldots$, and $h = 6.6262 \cdot 10^{-27}$ erg · s is the *Planck constant*. Then Planck named the product $h\nu$ the *energy element* and supposed that a given *excited* resonator may possess $n \geqslant 0$ of these elements.

As a consequence, any configuration $\mathcal{C}$ of the system of N excited/non-excited resonators will have a total energy $M_{\mathcal{C}}\,h\nu$ proportional to the *energy element* $h\nu$, which is also known as the (single) *light-quantum* energy (A. Einstein (1905) [5]).

Seeing that the cavity of the oven is *thermal* and, in order to establish contact with the temperature of radiation, Planck appealed, in the next step, to Statistical Mechanics. To this end he first calculated the *statistical weight* $\Gamma_N(M)$ of configurations in the system of N resonators for a *fixed* energy $E = M\,h\nu$, i.e., in the *microcanonical* ensemble, see, for example, [6, pp. 603–604]. Note that in fact the number $\Gamma_N(M)$ of distributions of M light-quanta over N resonators coincides with the number of possibilities of distributing M *objects* into N *boxes*:

$$\Gamma_N(M) = \frac{(M+N-1)!}{M!(N-1)!}. \tag{1}$$

Applying the *Stirling formula* for large N and M in (1), Planck then followed the *Boltzmann principle* for deducing the asymptotic form of the microcanonical *entropy*

$$S(M,N) := \ln \Gamma_N(M) = (M+N)\ln(M+N) - M\ln M - N\ln N, \tag{2}$$

and the temperature T corresponding to the system of N resonators: $1/(k_B T) := \partial_E S(E/h\nu, N)$, where $k_B = 1.38 \cdot 10^{-16}$ erg/K is the *Boltzmann constant*. Then (2) yields

$$\frac{1}{k_B T} = \partial_E \ln \Gamma_N(M) = \frac{1}{h\nu} \ln\left(1 + \frac{N}{M}\right). \tag{3}$$

Next Planck defined by $\varepsilon(\nu) = E/N$ the *mean-value* energy for resonators in the system of N excited/non-excited resonators and deduced from (3) that

$$\varepsilon(\nu) = \frac{h\nu}{e^{\beta\, h\nu} - 1}. \tag{4}$$

Finally, multiplying (4) by the *density number* of radiation modes *per* unit volume and *per* infinitesimal band, taken from the *classical* electrodynamics, he obtained

$$\rho(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{\beta\, h\nu} - 1}, \quad \beta = \frac{1}{k_B T}, \tag{5}$$

which is the *spectral density* of radiation emitted by a black-body at a given temperature T, that is, the *Planck formula* (1900).

We note that Planck did not attribute any definite physical significance to his hypothesis of *resonators* but rather proposed it as a mathematical *device* that enabled him to derive an expression for the black-body spectrum that matched the empirical data for all frequencies ν. Although initially he was inclined to consider the *light-quanta* of resonators only for exchanging the energy between radiation and the oven walls, at the end of the paper [4] Planck tentatively mentioned the possible connection of such *resonators* with a mono-atomic gas.

Much later Einstein (1916) [7] provided a different demonstration of the Planck formula. His arguments involved the idea of the *light-quanta* $h\nu$, but also their *interactions* with a gas of *two-level molecules* (with the level spacing according to the Bohr rule: $E_2 - E_1 = h\nu$) occupying the cavity, which has temperature T, see, e.g., [6, pp. 604–607] for details.

1.1.2. S. N. Bose and Bose–Einstein statistics

Recall that Einstein (1905) succeeded in explaining the *photoelectric emission* because he supposed that the light "*...consists of a finite number of energy quanta localized in space, which move without being divided and which can be absorbed or emitted only as a whole*" [5]. Moreover, the *Compton effect* [8] (1923) also clearly indicated that radiation *itself* consists of *light-quanta* (they were named *photons* by G. N. Lewis [9] in 1926). But the mathematically satisfactory quantum theory of the *light* considered as a *many-body* quantum system, with elucidation of the black-body radiation and the Planck law, was developed by Satyendra Nath Bose [10] only in 1924.

Originally Bose submitted his manuscript to the *Philosophical Magazine* (Taylor & Francis), but it was rejected there. He then sent it to Einstein with the humble request: "... If you think the paper is worthy of publication, I shall be grateful if you arrange its publication in *Zeitschrift für Physik*". In a footnote to the paper translated into German, Einstein wrote: "In my opinion Bose's derivation signifies an important advance. The method used here gives the quantum theory of the ideal gas (that is, of atoms, or molecules — *remark by VAZ*), as I will explain elsewhere".

In his pioneering paper [10], Bose *extended* Planck's method of quantisation of "imaginary" vibrating resonators, which by energy quanta directs the connection between radiation and matter (the oven walls, or Planck's "speck of carbon" [11]) to the *quantisation* of the electromagnetic field in cavity, which implies a non-classical *corpuscular* nature of radiation *itself* in the spirit of the light-quanta [5]!

To this end he started in [10] with a *plain-spoken* declaration that *light-quantum* (photon) with energy $\varepsilon = h\nu$ has a *momentum* $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$ of the magnitude $p = h\nu/c$ in *direction* of its movement.

Next, Bose proceeded with calculation in cavity of a *density of states* for photon-momentum operator $\widehat{p} = (\hbar/i)\nabla$, where $\hbar = h/2\pi$. For a cubic cavity, $\Lambda = L \times L \times L \subset \mathbb{R}^3$, with *periodic boundary* conditions one gets explicitly the eigenfunctions $\psi_{\mathbf{k}}(\mathbf{x}) = \exp(i\,\mathbf{k}\cdot\mathbf{x})$ and the spectrum $\sigma(\widehat{p})$ of the self-adjoint operator $\widehat{p}$ in a standard for the quantum mechanics Hilbert space $\mathcal{H} = L^2(\Lambda)$ of the square-integrable complex wave-functions in $\Lambda \subset \mathbb{R}^3$. They are enumerated by wave-vectors $\mathbf{k} = (k_x, k_y, k_z) \in \sigma(\widehat{p})/\hbar$:

$$\widehat{p}\,\psi_{\mathbf{k}} = \hbar\,\mathbf{k}\,\psi_{\mathbf{k}} \;\text{ and }\; \mathbf{k} \in \{2\pi/L\,(s_x, s_y, s_z) : (s_x, s_y, s_z) \in \mathbb{Z}^3\} =: \Lambda^*. \tag{6}$$

Then the number of the photon states in the infinitesimal band $[\nu, \nu + d\nu]$ is the number dN of eigenvectors in the volume of corresponding spherical shell $[k, k + dk]$ divided by the *k-lattice volume* per point (6). Since $p = \hbar k = h\nu/c$, one gets

$$dN(k) = \frac{4\pi k^2 dk}{(2\pi/L)^3} = L^3\,\frac{4\pi\nu^2 d\nu}{c^3}. \tag{7}$$

Now taking into account *two* states of *polarisation* of light (the *photon spin* orientations either parallel or antiparallel to its direction of motion), Bose deduced from (7) for density of the photon states (modes), per unit volume of cavity and per infinitesimal band, the expression

$$J(\nu) := \frac{8\pi\nu^2}{c^3}. \tag{8}$$

The value of (8) coincides with the first factor in the Planck formula (5) but with a new meaning: it is the one-particle photon density of states in the mode ν for the unit volume of cavity.

Finally, Bose considered photons $h\nu$ as identical *indistinguishable* particles allowed (similar to the Planck *energy elements*) that *finitely* many of them may accumulate in a single photon state with wave-vector $\mathbf{k}$. Therefore, any configuration of photons in cavity can be labelled by *occupation numbers* $\{N_{\mathbf{k}} = 0, 1, 2, \ldots\}_{\mathbf{k}\in\Lambda^*}$. With this *prescription* for counting the photon configurations in a hot cavity with photon density of states (8) and at the temperature T, Bose proved in [10] the Planck formula (5) for spectral density of the black-body radiation emitted by cavity. To this end (similarly to Planck) he applied in the last part of [10] the entropy variational principle of Statistical Mechanics for states in equilibrium, cf. [6, Ch. II, §4].

Bose's very natural *ansatz* about *indistinguishability* of photons ("bosons") turned out to be far-reaching. His *receipt* for counting the allowed configurations of many-body photon system was then extended by Einstein [12] to the *mono-atomic ideal* quantum gas. Here instead of momentum operator $\widehat{p} = (\hbar/i)\nabla$ (6) one considers for the gas of atoms with mass m in box Λ the one-particle *kinetic-energy* operator $T_\Lambda = \widehat{p}^2/2m$. For periodic boundary conditions on $\partial\Lambda$ it has eigenvalues/eigenfunctions:

$$T_\Lambda\, \psi_{\mathbf{k}} = \varepsilon_{\mathbf{k}}\psi_{\mathbf{k}} \ \text{ and } \ \mathbf{k} \in \{2\pi/L(s_x, s_y, s_z) : \, (s_x, s_y, s_z) \in \mathbb{Z}^3\} = \Lambda^*, \tag{9}$$

with eigenvalues $\varepsilon_{\mathbf{k}} := (\hbar\,\mathbf{k})^2/2m$ for eigenfunctions $\{\psi_{\mathbf{k}}(x) = \exp(i\,\mathbf{k}\cdot\mathbf{x})\}_{\mathbf{k}\in\Lambda^*}$. This passage from photons to the quantum ("boson") gas of atoms evidently modifies in Λ the one-particle density of states, but not the prescription (*statistics*) for counting of allowed configurations of many-body particle system since by virtue of the *indistinguishability* they can still be labelled only by *occupation numbers* $\{N_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}$ of the one-body vector states $\{\psi_{\mathbf{k}}(x)\}_{\mathbf{k}\in\Lambda^*}$.

Note that, owing to the Schrödinger equation for the ideal quantum gas in Λ, the N-particle eigenfunctions $\Psi_l^{(N)}(x_1, \ldots, x_N) = \Psi^{(N)}_{\{\mathbf{k}_j : 1\leqslant j\leqslant N\}}(x_1, \ldots, x_N)$, for N-particle operator $T_\Lambda^{(N)} := \sum_{1\leqslant j\leqslant N} \widehat{p}_j^{\,2}/2m$, can be presented as linear combinations of products:

$$\prod_{j=1}^{N} \psi_{\mathbf{k}_j}(x_j) = \prod_{\mathbf{k}\in\Lambda^*} \prod_{j(k)=1}^{N_{\mathbf{k}}} \psi_{\mathbf{k}}(x_{j(\mathbf{k})}) = \Phi^{(N)}_{\{N_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}}(x_1, \ldots, x_N). \tag{10}$$

Hence, the occupation number $N_{\mathbf{k}}$ coincides with the number of *identical* wave-functions $\{\psi_{\mathbf{k}}(x_{j(\mathbf{k})})\}$ in the left-hand side product of identity (10). This occupation number has upper bound: $N_{\mathbf{k}} \leqslant N$, and for $N_{\mathbf{k}} = N$ the *symmetric* function $\prod_{1\leqslant j\leqslant N} \psi_{\mathbf{k}}(x_j)$ in (10) describes N *indistinguishable* particles in the box Λ that occupied a single mode with $\mathbf{k} \in \Lambda^*$. To keep *indistinguishability* for general configurations of occupation numbers $\{N_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}$, one has to *symmetrise* the functions in (10). Therefore, the idea of the *boson* quantum systems, or *Bose–Einstein statistics*, was born in 1924 thanks to the papers [10] and [12].

Now we exploit this idea to elucidate that the concept of *indistinguishability* of N identical quantum particles has an important consequence due to the quasi-classical *thought* experiments concerning a permutation, for example, of two of them. In fact, this experiment implies that, being a unique solution of the Schrödinger equation, the *normalised* wave-function for indistinguishable particle stays (up to a phase factor $e^{i\phi}$) invariant with respect to permutation of the corresponding arguments of these particles:

$$\Psi_N(x_1, \ldots, x_s, \ldots, x_r, \ldots, x_N) = e^{i\phi}\Psi_N(x_1, \ldots, x_r, \ldots, x_s, \ldots, x_N). \tag{11}$$

Then because of (11) one gets after second permutation that $e^{2i\phi} = 1$, and consequently $e^{i\phi} = \pm 1$. For this reason the wave-functions of identical *indistinguishable* quantum particles can be only of two categories: *symmetric*, corresponding to the *Bose–Einstein* statistics (1924) for *bosons*, or *antisymmetric*, corresponding to the *Fermi–Dirac* statistics (1926) for *fermions*.

These ("non-local") *collective* properties of the quantum statistics for non-interacting *indistinguishable* particles conflict fundamentally with the *Boltzmann statistics*, which ensures a *statistical independence* of non-interacting classical particles. We clarify that in the case of (Bose–Einstein or Fermi–Dirac) quantum statistics for *indistinguishable* particles a list of occupation numbers $\{N_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}$ defines one and *unique* state $\Psi^{(N)}_{\{N_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}}$ of the quantum system. Whereas, if particles are *classical*, one has to *enumerate* them. Then besides the list $\{N_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}$ their classical *microstates* (configurations) depend on distribution of attributed *labels*. For the partition function this produces a supplementary degeneracy factor $N!/\prod_k N_{\mathbf{k}}!$, which yields the *Boltzmann counting* of allowed configurations and thus the *quasi-classical* limit of quantum system corresponding to *rarified* classical ideal gases. For details see, for example, [6, Ch. III, § 1].

Next, by analysis of *correlations* we strengthen the key observation that in contrast to the *statistical independence* (in the standard of this term) of particles of the classical ideal gas, the particles of an ideal quantum gas are *indistinguishable*, but *not* independent. For this, one can consider *two-point correlation functions* for (*spinless*) bosons $F_B(R)$ and fermions $F_F(R)$ separated by inter-particle distance $R = |\mathbf{r}_1 - \mathbf{r}_2|$. Then explicit calculations (see [13, Ch. 1, § 4, **e**)]) for N-body ideal Bose (+) and Fermi (−) gases in volume $V = |\Lambda|$ yield

$$F_{B/F}(R) = 1 \pm v^2 \left| \frac{1}{V} \sum_{\mathbf{k}\in\Lambda^*} n_k^{B/F} \, e^{i\,\mathbf{k}\cdot\mathbf{R}} \right|^2, \quad v = V/N. \tag{12}$$

Here the *grand-canonical* Gibbs expectation (*mean value*) of the occupation number N_k for bosons is $n_k^B = (e^{\beta(\varepsilon_k - \mu)} - 1)^{-1}$, whereas for fermions it is $n_k^F = (e^{\beta(\varepsilon_k - \mu)} + 1)^{-1}$, and μ is the value of the chemical potential in the grand-canonical ensemble. Then *out* from the quantum *degenerate* regime (for small densities, high temperatures $\theta = k_B T$, that is, when the *thermal* de Broglie wave-lengths, $\lambda_{\rm deB}(\theta) = \hbar/\sqrt{m\,\theta}$, are much *smaller* than the average *inter-particle distance* $\sqrt[3]{v}$), we obtain from (12) that, already within the first quantum corrections to the case of the *Boltzmann statistics*: $F(R) = 1$, the quantum correlations do exist:

$$F_{B/F}(R) \simeq 1 \pm e^{-(m\theta/\hbar^2)\,R^2}, \quad \hbar/\sqrt{m\,\theta} \ll \sqrt[3]{v}. \tag{13}$$

The interpretation of formula (13) is straightforward:

— Since for finite distances the correlation $F_B(R) > 1$, the bosons are affected by a *temperature-dependent* statistical *attraction* to each other. It is monotonously decreasing to non-correlation for ideal gas in the *Boltzmann regime*: $F(R) = 1$, for growing R.

— Since for finite distances the correlation $F_F(R) < 1$, the fermions are affected by a *temperature-dependent* statistical *repulsion* from each other with the same behaviour as for bosons for growing R, but (to bolster the *Pauli exclusion principle*) with a strong *repulsion* for $R \downarrow 0$.

In paper [14] the authors transformed these quantum *temperature-dependent* statistical correlations into particle *two-body potential* to treat the problem classically. In fact, the classical (i.e., the *Boltzmann*) limit of (12) (and thus (13)) corresponds to the high-temperature limit $\theta \to \infty$.

Note that in both cases of (13) the *effective radius* of correlation $R_{\rm corr}$ has order of the *thermal* de Broglie wave-lengths: $R_{\rm corr} \sim \lambda_{\rm deB}(\theta)$. For the quantum *degenerate* regime, that is, when the *thermal* de Broglie wave-lengths $\lambda_{\rm deB}$ are comparable to or larger than the average *inter-particle distance* $\sqrt[3]{v}$, the calculations yield for (12) qualitatively the same R-behaviour as for (13), but with a larger *effective radius* of correlations. A peculiarity that concerns the Bose gas is the limit (see [13, Ch. 1, §4, **e**)]):

$$\lim_{R\to 0} F_B(R) = 2 - \left(\frac{\langle N_0\rangle(\theta)}{N}\right)^2, \quad \hbar/\sqrt{m\,\theta} \geqslant \sqrt[3]{v}, \tag{14}$$

where $\langle N_0\rangle(\theta)$ is the Gibbs expectations (*mean value*) of the occupation number in the mode $\mathbf{k} = 0$, see (9). If this value is *macroscopic*, that is, $\lim_{N\to\infty} \langle N_0\rangle(\theta)/N = v\,\rho_0(\beta) > 0$ for $\beta = 1/\theta$, then $\rho_0(\beta)$ is density of the *Bose–Einstein condensate*, which will be the main subject of the next subsection.

1.2. Bose–Einstein condensation

1.2.1. Conventional Bose–Einstein condensation and G. E. Uhlenbeck's objections

Based on [12] (1924), Einstein applied then in [1] (1925) the ideas of Bose–Einstein statistics to study the thermodynamic properties of the *Ideal Bose gas* (IBG) and predicted in this system a peculiar *condensation* of particles. This phenomenon occurs in quantum *degenerate* regime and manifests itself as a *macroscopic* (proportional to the volume of the system) mean value of occupation number in *one* of the modes. But two years later, in his doctoral thesis "On statistical methods in the quantum theory" (Leiden, 1927) [15], G. E. Uhlenbeck criticised Einstein's arguments in favour of condensation on the mathematical ground. In particular, his critical remarks concern the quantisation in finite volume, the implementation of the thermodynamic limit and the accuracy with the replacement of certain sums by integrals, see [15, pp. 69–71]. We return to details of Uhlenbeck's objections below.

This criticism delayed the general acceptance of this *conventional* one-mode *Bose–Einstein condensation* (BEC) for almost a decade. It is discovery of *superfluidity* of liquid ^{4}He with λ-point phase transition at $T = 2.172$ K (for pressure 1 atm) by P. L. Kapitza (1938) [17], and J. F. Allen and A. D. Misener (1938) [18] that renewed interest to the BEC. For example, in [2] (1938) F. London wrote: "In discussing some properties of liquid helium, I recently realized that Einstein's statement had been erroneously discredited; moreover, some support could be given to the idea that the peculiar phase transition ("λ-point") ... very probably has to be regarded as the condensation phenomenon of the *Bose–Einstein statistics*, ..."

The paper [2] answered the objections formulated in [15, pp. 69–71] and elucidated the formal mathematical origin of the conventional one-mode BEC. In paper [16] Uhlenbeck withdrew his objections. In fact, the arguments presented by London were similar to the modern "finite-size scaling" approach to analysis of the phase transitions. To proceed, we demonstrate below his line of reasoning, which is now widely accepted. Subsequently it also provided a *generalisation* of the conventional concept of the BEC for ideal and interacting boson systems, see comments in Subsubsection 1.2.2 (see [19, 22]) and in Subsubsection 1.2.3 (cf. [23–25]).

With this aim in view, let N-particle ideal Bose gas be enclosed in the *cubic* box $\Lambda = L \times L \times L \subset \mathbb{R}^3$, $|\Lambda| = V$. We consider a possibility of the conventional BEC of the IBG in the thermodynamic limit $V \to \infty$. To this end (similarly to Subsubsection 1.1.2) we consider in Hilbert space $\mathcal{H} = L^2(\Lambda)$ the self-adjoint extension of the one-particle kinetic-energy Hamiltonian

$$T_\Lambda := \left(-\frac{\hbar^2}{2m}\Delta\right)_{\Lambda,\text{p.b.}}, \tag{15}$$

with *periodic boundary* (p.b.) conditions on $\partial\Lambda$. Then the one-particle spectrum is $\sigma(t_\Lambda) := \{\varepsilon_{\mathbf{k}} := \hbar^2\mathbf{k}^2/(2m)\}_{\mathbf{k}\in\Lambda^*}$, where m denotes the mass of the particle and

$$\Lambda^* := \{k_j = 2\pi n_j/L : n_j \in \mathbb{Z}\}_{j=1}^3 \tag{16}$$

is a *dual* to Λ (with respect to p.b. conditions) set of wave-vectors. Recall that, by virtue of (9), (10) and due to the Bose–Einstein statistics, the symmetric *eigenfunctions* $\{\Psi_l^{(N)}\}_{l\geqslant 1}$ of the self-adjoint kinetic-energy Hamiltonian $T_\Lambda^{(N)}$ for N-particle ideal Bose gas are entirely and uniquely determined by *configurations* of occupation numbers $\{N_{\mathbf{k}} \geqslant 0\}_{\mathbf{k}\in\Lambda^*}$ in modes $\mathbf{k} \in \Lambda^*$ for corresponding functions $\Psi^{(N)}_{\{N_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}}$, such that

$$T_\Lambda^{(N)}\Psi^{(N)}_{\{N_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}} = \sum_{\mathbf{k}\in\Lambda^*} \varepsilon_{\mathbf{k}} N_{\mathbf{k}}\, \Psi^{(N)}_{\{N_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}}. \tag{17}$$

This bijection reduces the quantum Gibbs calculations for *canonical ensemble* (θ, V, N) to Statistical Mechanics on *configurations* of occupation numbers (cf. (17)) for *canonical* partition function:

$$Z_\Lambda(\theta, V, N) = \mathrm{Tr}_{\mathcal{H}^N}\,(e^{-\beta\, T_\Lambda^{(N)}}) = \sum_{\{N_{\mathbf{k}}\geqslant 0\}_{\mathbf{k}\in\Lambda^*}} e^{-\beta\sum_{\mathbf{k}\in\Lambda^*}\varepsilon_{\mathbf{k}} N_{\mathbf{k}}}\, \delta(N, \sum_{\mathbf{k}\in\Lambda^*} N_{\mathbf{k}})\,, \quad \beta = 1/\theta. \tag{18}$$

Here $\delta(x,y)$ for $x,y \in \mathbb{N}_0$ is the *Kronecker* symbol. To escape this constraint and to profit of explicit calculations (as in [2]), we pass on to the *grand-canonical ensemble* (θ, V, μ), where the chemical potential $\mu < 0$ controls *total* number of particles in Λ. Then, the *grand-canonical* partition function gets the form

$$\Xi_\Lambda(\theta,V,\mu) = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_\Lambda(\theta,V,N) = \sum_{\{N_{\mathbf{k}}\geqslant 0\}_{\mathbf{k}\in\Lambda^*}} \prod_{\mathbf{k}\in\Lambda^*} e^{-\beta(\varepsilon_{\mathbf{k}}-\mu)N_{\mathbf{k}}} = \prod_{\mathbf{k}\in\Lambda^*} \sum_{N_{\mathbf{k}}=0}^{\infty} e^{-\beta(\varepsilon_{\mathbf{k}}-\mu)N_{\mathbf{k}}}. \quad (19)$$

As a consequence of (19), the *grand-canonical* Gibbs probability distribution is a *product-measure* such that for expectations (*mean values*) $n_q(\beta,\mu) := \langle N_q\rangle_{T_\Lambda}(\beta,\mu)$ of occupation numbers $N_{\mathbf{q}}$ in any mode $\mathbf{q}\in\Lambda^*$ one obtains

$$\langle N_q\rangle_{T_\Lambda}(\beta,\mu) := \frac{1}{\Xi_\Lambda(\theta,V,\mu)} \prod_{\mathbf{k}\in\Lambda^*\setminus q} \sum_{N_{\mathbf{k}}=0}^{\infty} e^{-\beta(\varepsilon_{\mathbf{k}}-\mu)N_{\mathbf{k}}} \sum_{N_{\mathbf{q}}=0}^{\infty} e^{-\beta(\varepsilon_{\mathbf{q}}-\mu)N_{\mathbf{q}}} N_{\mathbf{q}} = \frac{1}{e^{\beta(\varepsilon_{\mathbf{q}}-\mu)}-1}, \quad (20)$$

where for ideal *bosons* $\mu < 0$, since $\varepsilon_k = \hbar^2\mathbf{k}^2/(2m) \geqslant 0$, $\mathbf{k}\in\Lambda^*$, by virtue of (16).

Owing to (16) and (20), the grand-canonical expectation of the *total* density of bosons in Λ is

$$\rho_\Lambda(\beta,\mu) := \frac{1}{V}\sum_{\mathbf{k}\in\Lambda^*} \frac{1}{e^{\beta(\varepsilon_{\mathbf{k}}-\mu)}-1} = \frac{1}{L^3} \sum_{\{n_j\in\ \mathbb{Z}:\, j=1,2,3\}} \left\{ e^{\beta(\hbar^2\sum_{j=1}^{3}(2\pi n_j/L)^2/2m-\mu)} - 1\right\}^{-1}. \quad (21)$$

(a) To study the values of the chemical potential for observables, one first considers the *equation* for μ and a given total particle density ρ in a *finite* volume $V = |\Lambda|$:

$$\rho = \rho_\Lambda(\beta,\mu), \quad \mu < 0. \quad (22)$$

Seeing that due to the term $\{\mathbf{k}=0\}$ in (21) the limit $\lim_{\mu\to 0}\rho_\Lambda(\beta,\mu) = +\infty$, the solution $\mu_\Lambda(\beta,\rho)$ of equation (22) *always* exists and $\mu_\Lambda(\beta,\rho) < 0$ for $\rho \geqslant 0$. As a consequence, there is no *macroscopic* (proportional to the volume) occupation of any single mode $\mathbf{k}\in\Lambda^*$, see (20) for $\mu = \mu_\Lambda(\beta,\rho)$, and, thus (as was noticed by Uhlenbeck in [15]) there is *no trace* of the BEC or of any phase transitions.

At that time this conclusion was not anymore surprising [16], since after the "100th Anniversary of Van der Waals' Birthday Congress" (Amsterdam, 1937) the idea that condensation as a phase transition could mathematically be understood only in the *thermodynamic limit* $V\to\infty$ became dominating.

(b) Let $\mu < 0$ and $\Lambda \nearrow \mathbb{R}^{d=3}$. Note that the last term in (21) is nothing but the integral *Darboux–Riemann* sum, which converges for $L\to\infty$ to integral $\mathfrak{I}_{d=3}(\beta,\mu)$:

$$\rho(\beta,\mu) := \lim_{\Lambda}\rho_\Lambda(\beta,\mu) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k \left\{ e^{\beta(\hbar^2k^2/2m-\mu)} - 1\right\}^{-1} =: \mathfrak{I}_{d=3}(\beta,\mu). \quad (23)$$

Note that for $d > 2$ the integral $\mathfrak{I}_d(\beta,\mu)$ is convergent and bounded for $\mu \leqslant 0$ with $\sup_{\mu\leqslant 0}\mathfrak{I}_{d=3}(\beta,\mu) = \rho(\beta,\mu=0) =: \rho_c(\beta)$. Hence we face up to the fact that in the *thermodynamic limit* equation (22) gets the form $\rho = \rho(\beta,\mu)$, $\mu\leqslant 0$, and has solutions $\mu(\beta,\rho)$ *only* for densities less than the critical density: $\rho \leqslant \rho_c(\beta)$. This observation, which looks as a *defect* of the model, called the *ideal* Bose gas [6, Ch. III], has been translated by Uhlenbeck in [15] as a *no-go* assertion about possibility of the BEC phase transition even after the thermodynamic limit.

(c) An elegant way to resolve the paradox in (b) and to obtain BEC was suggested by London [2] (1938). Formally his arguments could be presented as follows. If we search for solutions of equation (22) for $\mu < 0$ in the limit $V \to \infty$, then the mathematical problem is to describe a *family* of solutions of (22) for the sequence of functions $\{\mu \mapsto \rho_\Lambda(\beta, \mu)\}_\Lambda$ *unbounded* for $\mu \in (-\infty, 0)$, which converges *non-uniformly* in μ to a function $\rho(\beta, \mu)$ *bounded* in $(-\infty, 0)$.

Since a *singularity* preventing the uniform convergence is due to the $\{\mathbf{k} = 0\}$-term in the right-hand side of (21), we re-write equation (22) as follows:

$$\rho = \frac{1}{L^3}\left\{e^{-\beta\mu} - 1\right\}^{-1} + \frac{1}{L^3}\sum_{\{n_j \in \mathbb{Z}: j=1,2,3\}\setminus\{\mathbf{0}\}} \left\{e^{\beta(\hbar^2\sum_{j=1}^3 (2\pi n_j/L)^2/2m - \mu)} - 1\right\}^{-1}. \tag{24}$$

It is important to remark that for $L \to \infty$ the *Darboux–Riemann* sum in the right-hand side of (24) converges to integral $\mathfrak{I}_{d>2}(\beta, \mu) \leqslant \rho_c(\beta)$ (23) *uniformly* in $\mu \leqslant 0$.

(I) Case $\rho < \rho_c(\beta)$

Then the *Darboux–Riemann* sum in the right-hand side of (24) is also less than $\rho_c(\beta)$ where solutions of equation (24) are $\mu_\Lambda(\beta, \rho) < 0$. Seeing that for $L \to \infty$ this sum converges uniformly in $\mu < 0$ to integral (23), which is less than $\rho_c(\beta)$, we deduce that the limit $\lim_{L\to\infty} \mu_\Lambda(\beta, \rho) = \mu(\beta, \rho) < 0$. Therefore, this strictly negative limit is solution for (23): $\rho = \rho(\beta, \mu(\beta, \rho)$, whereas for another term in (24) we obtain $\lim_{L\to\infty} L^{-3}(\exp(-\beta\mu_\Lambda(\beta, \rho)) - 1)^{-1} = 0$.

(II) Case $\rho > \rho_c(\beta)$

For these values of the total density ρ, the volume-dependent *family* of solutions $\{\mu_\Lambda(\beta, \rho)\}_\Lambda$ of equation (24) satisfies the identity

$$\rho - \frac{1}{L^3}\sum_{\{n_j \in \mathbb{Z}: j=1,2,3\}\setminus\{\mathbf{0}\}} \left\{e^{\beta(\hbar^2\sum_{j=1}^3 (2\pi n_j/L)^2/2m - \mu_\Lambda(\beta,\rho))} - 1\right\}^{-1} = \frac{1}{L^3}\left\{e^{-\beta\mu_\Lambda(\beta,\rho)} - 1\right\}^{-1}. \tag{25}$$

Owing to the facts that (a) the *Darboux–Riemann* sum in the right-hand side of (24) for $\mu \leqslant 0$ is less than $\rho_c(\beta)$ and (b) for $L \to \infty$ it converges to integral $\mathfrak{I}_{d>2}(\beta, \mu) \leqslant \rho_c(\beta)$ (23) *uniformly* in $\mu \leqslant 0$, we deduce the limits:

$$\lim_{L\to\infty}\frac{1}{L^3}\sum_{\{n_j \in \mathbb{Z}: j=1,2,3\}\setminus\{\mathbf{0}\}} \left\{e^{\beta(\hbar^2\sum_{j=1}^3 (2\pi n_j/L)^2/2m - \mu_\Lambda(\beta,\rho))} - 1\right\}^{-1} = \\ = \mathfrak{I}_{d>2}(\beta, \lim_{L\to\infty}\mu_\Lambda(\beta, \rho)) \leqslant \rho_c(\beta). \tag{26}$$

Then by condition $\rho > \rho_c(\beta)$ and equations (25) and (26), one gets that solutions $\{\mu_\Lambda(\beta, \rho)\}_\Lambda$, for $L \to \infty$, *must* converge to *zero* and that

$$\rho - \rho_c(\beta) = \lim_{L\to\infty}\frac{1}{L^3}\left\{e^{-\beta\mu_\Lambda(\beta,\rho)} - 1\right\}^{-1} > 0. \tag{27}$$

— As a consequence, (27) yields that for $\rho > \rho_c(\beta)$ there is a *macroscopic* occupation of the mode $\{\mathbf{k} = 0\}$:

$$\rho_0(\beta) := \lim_{V\to\infty}\frac{\langle N_{k=0}\rangle_{T_\Lambda}(\beta, \mu_\Lambda(\beta, \rho))}{V} = \rho - \rho_c(\beta) > 0, \tag{28}$$

that is, the Bose–Einstein condensation at *zero-momentum mode.*

— Moreover, (27) allows one to estimate the *rate* of convergence of solutions $\{\mu_\Lambda(\beta,\rho)\}_\Lambda$ to zero:

$$\mu_\Lambda(\beta,\rho>\rho_c(\beta)) = -\frac{1}{L^3}\,\frac{1}{\beta(\rho-\rho_c(\beta))} + o(1/V), \quad L\to\infty. \tag{29}$$

Taking into account (29), one can check the occupation density of *non-zero* modes. By virtue of $\{\mathbf{k}=(2\pi/L)\{n_1,n_2,n_3\}\neq\mathbf{0}\}$ and by asymptotics (29), we obtain

$$\lim_{V\to\infty}\frac{\langle N_{\mathbf{k}}\rangle_{T_\Lambda}(\beta,\mu_\Lambda(\beta,\rho))}{V} =$$

$$\lim_{L\to\infty}\frac{1}{L^3}\left\{\exp\left(\beta\left(\hbar^2\sum_{j=1}^{3}(2\pi n_j/L)^2/2m-\mu_\Lambda(\beta,\rho)\right)\right)-1\right\}^{-1} = \tag{30}$$

$$\lim_{L\to\infty}\frac{1}{L^3}\left\{\exp\left(\beta\left(\hbar^2\sum_{j=1}^{3}(2\pi n_j/L)^2/2m+\,L^{-3}\left(\beta(\rho-\rho_c(\beta))\right)^{-1}-o(1/L^3)\right)\right)-1\right\}^{-1} =$$

$$\lim_{L\to\infty}\frac{1}{L^3}\left\{\exp\left(\beta\hbar^2\sum_{j=1}^{3}(2\pi n_j/L)^2/2m\right)-1\right\}^{-1} = 0. \tag{31}$$

Hence, for *any* density of bosons ρ there is no BEC in *any* of *non-zero* modes $\{\mathbf{k}\neq\mathbf{0}\}$.

(III) Case $\rho=\rho_c(\beta)$

The analysis of this case is more delicate [25, Ch. 2]. Instead of (29) one gets asymptotics $\mu_\Lambda(\beta,\rho_c(\beta))\simeq\mathcal{O}(1/V^\alpha)$, for some $\alpha\in(2/3,1)$.

Summarising, the ideal Bose gas in *cubic* box $\Lambda=L\times L\times L\subset\mathbb{R}^{d=3}$, for a given temperature and for particle densities larger than critical, manifests the one-mode Bose–Einstein condensation (28) in the ground zero-mode state for one-particle kinetic-energy operator with periodic self-adjoint extension. There are straightforward generalisations of this assertion to dimensions $d>2$, as well as to any bounded convex $\Lambda\subset\mathbb{R}^d$ with smooth boundary $\partial\Lambda$ and to different self-adjoint extensions with *attractive/non-attractive* boundary conditions, see [26, 27].

After 1938 new developments in the theory of BEC in continuous *translation-invariant* systems emerged: the first one in 1982–1986, due to van den Berg–Lewis–Pulè (a *generalised* BEC of Types I, II, III), the second one a bit later, when van den Berg, Lewis and Pulè (1988) [23] discovered in the *Huang–Yang–Luttinger* (HYL) model [28] for *interacting* bosons a *non-conventional* BEC, which exists *because of* interaction. Later, such an unusual non-conventional BEC was also found in the *Bogoliubov weakly imperfect Bose gas* (1998) [29].

Nowadays the *one-mode* BEC (28) is known as the *conventional* Bose–Einstein condensation of Type I, see [22]. We shall return to the *generalised* BEC and to the *non-conventional* BEC in the next subsubsections.

1.2.2. Generalised (Bose–Einstein) condensation à la van den Berg–Lewis–Pulè

The simplest way to understand different types of *generalised* BEC is to consider the example motivated by *Hendrik Casimir* [20] (1968). Let us instead of the *cubic box* $\Lambda=L\times L\times L\subset\mathbb{R}^3$, $|\Lambda|=V$, take a *prism* $\Lambda=L_1\times L_2\times L_3$ of the *same* volume with the sides of length $L_j=V^{\alpha_j}$, $j=1,2,3$, such that $\alpha_1\geqslant\alpha_2\geqslant\alpha_3>0$ and $\alpha_1+\alpha_2+\alpha_3=1$. Here we ignore a conflict between *linear* and *volume* dimensionalities as irrelevant for further calculations.

1.2.2.1. Generalised BEC Type II

Let Λ be the *anisotropic Casimir* box, that is, $\alpha_1 = 1/2$. Since for $\{\mathbf{k} = (2\pi)\{n_1/V^{1/2}, 0, 0\}\}_{n_1\in\mathbb{Z}}$ (16) one gets $\varepsilon_k = (2\pi\hbar\, n_1)^2/2mV$, we re-write equation (24) as follows:

$$\rho = \frac{1}{V}\frac{1}{e^{-\beta\mu}-1} + \frac{1}{V}\sum_{k\in\{\Lambda^*:n_1\neq 0, n_2=n_3=0\}} \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} + \frac{1}{V}\sum_{k\in\{\Lambda^*:n_j\neq 0, j=2\text{or}3\}} \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}. \tag{32}$$

Note that for $\mu < 0$ the *second* term in the right-hand side of (32) is the *Darboux–Riemann* sum for *one-dimensional integral* divided by $V^{\alpha_2+\alpha_3}$ and, thus, tends to *zero* for $V\to\infty$. The *third* term in the right-hand side of (32) is the *Darboux–Riemann* sum for *three-dimensional integral* (23), which, as above, converges, for $V\to\infty$, to $\mathfrak{I}_3(\beta,\mu) \leqslant \rho_c(\beta)$ *uniformly* in $\mu \leqslant 0$. If $\{\mu_\Lambda(\beta,\rho)\}_\Lambda$ are solutions of equation (32), then it yields the limits:

$$\lim_{V\to\infty}\left\{\frac{1}{V}\frac{1}{e^{-\beta\mu_\Lambda(\beta,\rho)}-1} + \frac{1}{V}\sum_{k\in\{\Lambda^*:n_1\neq 0, n_2=n_3=0\}} \frac{1}{e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))}-1}\right\} = \\ = \rho - \lim_{V\to\infty}\mathfrak{I}_3(\beta,\mu_\Lambda(\beta,\rho)). \tag{33}$$

Note that if $\mu_\Lambda(\beta,\rho) < 0$, then the limit in the left-hand side of (33) is *zero*. Therefore, to satisfy (33) for $\rho > \rho_c(\beta)$ the solutions $\{\mu_\Lambda(\beta,\rho)\}_\Lambda$ of equation (32) *must* converge to zero: $\lim_{V\to\infty}\mu_\Lambda(\beta,\rho) = 0$. This may ensure that the limit in the left-hand side yields a positive difference: $\rho - \lim_{V\to\infty}\mathfrak{I}_3(\beta,\mu_\Lambda(\beta,\rho)) = \rho - \rho_c(\beta) > 0$. On account of $\alpha_1 = 1/2$ one obtains in the left-hand side of (33) that $\varepsilon_k = (2\pi\hbar\, n_1)^2/2mV$ for $\{\mathbf{k} = (2\pi)\{n_1/V^{1/2}, 0, 0\}\}_{n_1\in\mathbb{Z}}$. Hence, a non-zero limit in the left-hand side of (33) implies for solutions $\mu_\Lambda(\beta, \rho > \rho_c(\beta))$ the asymptotic behaviour:

$$\mu_\Lambda(\beta,\rho>\rho_c(\beta)) = -\frac{A}{V} + o\left(\frac{1}{V}\right), \quad A > 0. \tag{34}$$

Then asymptotics (34) suggests for the left-hand side of (33) the non-zero limit:

$$\lim_{V\to\infty}\frac{1}{V}\sum_{k\in\{\Lambda^*:(n_1,0,0)\}}\frac{1}{e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))}-1} = \frac{1}{\beta}\sum_{n_1=0,\pm1,\pm2,\dots}\left(\frac{(2\pi\hbar)^2}{2m}n_1^2 + A\right)^{-1}. \tag{35}$$

As a consequence, the limits (33), (35) provide a *generalised* BEC $\rho_0(\beta)$ with *macroscopic* occupation of a *countable* number of modes:

$$\rho_0(\beta) = \frac{1}{\beta}\sum_{n_1=0,\pm1,\pm2,\dots}\left(\frac{(2\pi\hbar)^2}{2m}n_1^2 + A\right)^{-1} = \rho - \rho_c(\beta). \tag{36}$$

Here parameter $A = A(\beta,\rho)$ (34) is a *unique* root of equation (36). Moreover, seeing that $\alpha_2+\alpha_3 = 1/2$, the limit (35) with ε_k for other modes $\{\Lambda^* : n_j \neq 0, j = 2,3\}$ is *nulle* because $\varepsilon_k \sim \mathcal{O}(V^{-2\alpha_2})$ or $\mathcal{O}(V^{-2\alpha_3})$, whereas $\mu_\Lambda(\beta,\rho>\rho_c(\beta)) \sim \mathcal{O}(V^{-1})$ (34). This shows that for $\rho > \rho_c(\beta)$ only modes $\{\Lambda^* : (n_1,0,0)\}$ are *macroscopically* occupied:

$$\lim_{V\to\infty}\frac{1}{V}\,\langle N_k\rangle_{T_\Lambda}(\beta,\mu_\Lambda(\beta,\rho)) = \\ = \left\{\begin{array}{ll}\beta^{-1}\left((2\pi\hbar n_1)^2/2m + A(\beta,\rho)\right)^{-1}, & \text{for } k\in\{\Lambda^*:(n_1,0,0)\} \\ 0, & \text{for } k\in\{\Lambda^*:n_j\neq 0, j=2,3\}\end{array}\right\}. \tag{37}$$

This means that for a *long* anisotropic prism with $\alpha_1 = 1/2$ in the *thermodynamic limit* $(L \to \infty)$ there exists *macroscopic* occupation of an *infinite* number of low-lying modes $k \in \{\Lambda^* : (n_1, 0, 0)\}$ including the *zero-mode* $k = 0$. In contrast to the *Type* I BEC, which occupies one, or few modes, this is the case of the van den Berg–Lewis–Pulè *generalised* BEC of *Type* II (1986) [22].

1.2.2.2. Generalised BEC Type III

Now let $\alpha_1 > 1/2$. That is, we consider a *highly anisotropic* prism still in one direction $j = 1$ [22]. Then

$$\inf_{k\in\Lambda^*\setminus\{0\}} \varepsilon_k = \frac{(2\pi\hbar)^2}{2m}\frac{1}{V^{2\alpha_1}}, \quad 2\alpha_1 > 1, \tag{38}$$

which corresponds to the mode with $(n_1 = 1, n_2 = 0, n_3 = 0)$. Since for any $\mu < 0$ the right-hand side of (32) converges to the integral $\rho(\beta, \mu)$ monotonously increasing up to $\rho_c(\beta)$ for $\mu \to -0$, the solution $\mu_\Lambda(\beta, \rho > \rho_c(\beta))$ of (32) has (for $V \to \infty$) the asymptotics:

$$\mu_\Lambda(\beta, \rho > \rho_c(\beta)) = -\frac{B}{V^\delta} + o\left(\frac{1}{V^\delta}\right), \quad B > 0, \quad \delta > 0. \tag{39}$$

For calculation of B and δ, we remark that the first *two* terms in the right-hand side of (32) may be represented as

$$\frac{1}{V}\sum_{k\in\{\Lambda^*:(n_1,0,0)\}} \frac{1}{e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))}-1} = \frac{1}{V}\sum_{k\in\{\Lambda^*:(n_1,0,0)\}} \sum_{s=1}^{+\infty} e^{-s\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))} =$$

$$= \frac{1}{V}\sum_{s=1}^{+\infty} e^{s\beta\mu_\Lambda(\beta,\rho)} \sum_{n_1=0,\pm1,\pm2,\ldots} e^{-s\beta\left((2\pi\hbar n_1)^2/2mV^{2\alpha_1}\right)}. \tag{40}$$

Notice that for the last sum in (40) the *Jacobi identity* gives

$$\sum_{n_1=0,\pm1,\pm2,\ldots} e^{-\pi\lambda n_1^2} = \frac{1}{\sqrt{\lambda}}\sum_{\xi=0,\pm1,\pm2,\ldots} e^{-\left(\pi\xi^2/\lambda\right)}, \tag{41}$$

where $\lambda = s\beta\, 2\pi\hbar^2 V^{-2\alpha_1}/m$. Taking into account (32) and (39)–(41), we find that for $\lambda \to 0$ only the term with $\xi = 0$ is important for (40) when $V \to \infty$, and the limit (33) takes the form

$$\rho - \rho_c(\beta) = \lim_{V\to\infty}\left\{\left(\frac{2\pi\hbar^2}{m}\beta\right)^{-1/2}\left\{\frac{V^{\alpha_1-1}}{V^{\delta/2}}\cdot V^\delta\right\}\frac{1}{V^\delta}\left\{\sum_{s=1}^{+\infty} e^{-\beta B\left(s/V^\delta\right)}\left(\frac{s}{V^\delta}\right)^{-1/2}\right\}\right\}. \tag{42}$$

By inspection the right-hand side of (42) with the *Darboux–Riemann* sum is *nontrivial* only for

$$\delta = 2(1-\alpha_1). \tag{43}$$

Then for $\rho - \rho_c(\beta) > 0$ one gets

$$\rho - \rho_c(\beta) = \left(\frac{2\pi\hbar^2}{m}\right)^{-1/2}\beta^{-1/2}\int_0^{+\infty} d\xi\, e^{-\beta B\xi}\xi^{-1/2}, \tag{44}$$

where $B = B(\beta, \rho) > 0$ is the *unique* root of equation (44), that is,

$$\rho - \rho_c(\beta) = \left(\frac{m}{2\hbar^2}\right)^{1/2} \frac{1}{\beta\sqrt{B(\beta,\rho)}}. \tag{45}$$

Thanks to (39), (43) and (45), we obtain for $\rho > \rho_c(\beta)$ and $V \to \infty$ the asymptotics

$$\varepsilon_k - \mu_\Lambda(\beta,\rho) \simeq \left\{ \begin{array}{l} (\hbar^2/2m)\,(2\pi n_1/V^{\alpha_1})^2 + (m/2\hbar^2)/(\beta^2(\rho-\rho_c(\beta))^2 V^{2(1-\alpha_1)}), \\ \ k \in \{\Lambda^* : (n_1,0,0)\} \\ (\hbar^2/2m)\left[(2\pi n_2/V^{\alpha_2})^2 + (2\pi n_3/V^{\alpha_3})^2\right] + \\ (m/2\hbar^2)/(\beta^2(\rho-\rho_c(\beta))^2 V^{2(1-\alpha_1)}),\ k \in \{\Lambda^* : n_{j=2 \text{ or } 3} \neq 0\} \end{array} \right\}. \tag{46}$$

Since $\alpha_1 > 1/2$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, the asymptotics (46) imply

$$\lim_{V\to\infty} \frac{1}{V} \langle N_k \rangle_{T_\Lambda}(\beta, \mu_\Lambda(\beta, \rho > \rho_c(\beta))) = 0,\ k \in \Lambda^*, \tag{47}$$

and at the same time

$$\lim_{V\to\infty} \frac{1}{V^{2(1-\alpha_1)}} \langle N_k \rangle_{T_\Lambda}(\beta, \mu_\Lambda(\beta, \rho > \rho_c(\beta))) = \frac{m/2\hbar^2}{\beta^2(\rho-\rho_c(\beta))^2},\ k \in \{\Lambda^* : (n_1,0,0)\}, \tag{48}$$

thought by (32), (40), (42), we deduce for density of *generalised* BEC

$$\rho_0(\beta) = \rho - \rho_c(\beta) = \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\{\Lambda^*:(n_1,0,0)\}} \langle N_k \rangle_{T_\Lambda}(\beta, \mu_\Lambda(\beta,\rho)) > 0. \tag{49}$$

This means that in the case of *extremely* long prism with $\alpha_1 > 1/2$ there exists for $\rho > \rho_c(\beta)$ a *conventional generalised* BEC with density $\rho_0(\beta) > 0$ (49), which is according to van den Berg–Lewis–Pulé BEC of *Type* III [22], because there is *no any* macroscopically occupied level in Λ^*, see (47) and (48).

These observations give a motivation for the following van den Berg–Lewis–Pulé's *classification* of the *generalised* BEC (gBEC) [19, 20, 22]:

- the condensation is called *Type I* if a *finite* number of single-particle levels are macroscopically occupied;
- it is of *Type II* if an *infinite* number of the levels are macroscopically occupied;
- it is called *Type III*, or the *non-extensive* condensation, if *none* of the levels is macroscopically occupied, whereas one has:

$$\rho_0(\beta) = \lim_{\delta\to 0^+} \lim_{V\to\infty} \frac{1}{V} \sum_{\{k\in\Lambda^*, 0\leqslant\|k\|\leqslant\delta\}} \langle N_k \rangle = \rho - \rho_c(\beta). \tag{50}$$

  This double limit (50) is the van den Berg–Lewis–Pulé *definition* of the condensed fraction of bosons, which includes all *Types* of BEC in modes $\{k \in \Lambda^*\}$, cf. (49).

After the Casimir example [20], van den Berg, Lewis and Pulè [21, 22] discovered that BEC in *exponentially* anisotropic boxes may produce new phenomena: the *second critical* density $\rho_m(\beta) > \rho_c(\beta)$ and a quite unusual transition between generalised condensations of *Type* I and *Type* III. This observation was then also confirmed for other types of *exponentially* anisotropic particle confinements in [30].

1.2.3. Non-conventional (Bose–Einstein) condensation

According to Subsubsections 1.2.1 and 1.2.2, the BEC of *Type* I (28), as well as the BEC of *Type* II (36) and of *Type* III (49), appears in the ideal Bose gas for $\rho > \rho_c(\beta)$ due to the *saturation* mechanism owing to the *finite* value of the *critical* density $\rho_c(\beta) < \infty$. (We remind that this terminology is related to phenomenon when saturated water vapor ("Bose gas") will condense to form liquid water called *dew* ("condensate"). It happens when density of vapor is growing to become *saturated* at the critical vapor density, known as the *dew-point*.)

Although the BEC, or *generalised* BEC (50) in the ideal Bose gas, was studied in great detail, analysis of condensate in the *interacting* Bose gas is a more delicate problem. Recall that *effective statistical* attraction between bosons (see Subsubsection 1.1.2), which is behind of the BEC for the ideal Bose gas, makes this system unstable with respect to any *direct attractive* interaction between bosons. So, efforts around the question "Why do interacting bosons condense?" were essentially concentrated around *repulsive* interactions between bosons. The studying of stability of the *conventional* BEC (or gBEC) in the non-ideal Bose gas with a rapidly decaying direct two-body *repulsive* interaction is still in progress, see, e.g., [31, 32]. On the other hand, if one counterbalances direct (*and statistical*) attractive interaction by a repulsion stabilising the boson system, this attraction may be the origin of a new (*non-saturating*) mechanism of BEC called the *non-conventional* (dynamical) condensation [34]. Implicitly the *non-conventional* condensation was considered for the first time in [23] on the basis of their rigorous study of condensation in the Huang–Yang–Luttinger (HYL) model [28].

We note that it was D. J. Thouless [35] who presented an instructive "back-of-the-envelope" calculations, which argue that a new kind of Bose condensation may occur in the HYL model of the hard-sphere Bose gas with a *jump* of the condensation density (as a function of the chemical potential) at the critical point. In [24] it was called the Thouless effect. Ten years later, the *non-conventional* condensation with a *jump* on the critical line (for *condensed–noncondensed* phases) was also discovered in the Bogoliubov Weakly Imperfect Bose Gas (WIBG), see [29] and reviews [25, 34].

The difference between *conventional* and *non-conventional* condensations reflects the difference in the mechanism of their formation [23, 24]:

— The *conventional* condensation is a consequence of the balance between *entropy* and *kinetic energy* via mechanism of saturation.

— The *non-conventional* condensation results from the balance between *entropy* and *interaction energy*, that is, via interaction-induced mechanism.

This difference has an important consequence: a non-conventional condensation occurs only *due* to particle interactions.

The last remark also motivates another name for non-conventional condensation: the *dynamical* condensation [33, 34]. As a consequence, the dynamical condensation may occur in *low-dimensional* ($d \geqslant 1$) boson systems, when there is *no* condensation *without* interaction, and it may exhibit the *first-order* phase transition with a *jump* in the density of condensate at the critical point (or line). Both the HYL and WIBG models manifest these properties, which for the HYL model have been predicted in [35] and then proved in [23, 24], see [36] for further development and more results in this area.

For the WIBG model, the proof of the jump in the density of the *zero-mode* condensate at the critical line between two phases (condensed–noncondensed) was demonstrated in [37] for dimensions $d \geqslant 1$. Besides the fact of condensation at *low* dimensions, the interaction-induced condensate in the WIBG model emerges in *two* stages. First, it appears as *non-conventional* zero-mode condensate with a *jump* (Thouless effect), and second, as a (continuously growing) *conventional* generalised Bose–Einstein condensate of *Type* I (*out* of the zero mode!), due to the particles *saturation* mechanism, see [38, Sections 5.2 and 5.3]. Note that for calculation of the above-mentioned condensate of *Type* I out of the zero mode, one must use the van den Berg–Lewis–Pulé definition (50), which is the way to take into account the modes involved in condensation, cf. [38, Corollary 5.17] and [25, Section 5].

In conclusion, it is instructive to examine the HYL model in more detail. First, we note that it has a Hamiltonian, which is diagonal in the occupation number operators, cf. the kinetic-energy operator (17). Then, it follows that, owing to the correspondence established in Subsubsection 1.2.1, it is possible to regard the occupation numbers $\{N_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}$ as *random* variables (with values in the natural numbers and zero) rather than as operators. For example, the variable corresponding to a total number of bosons in the box Λ is $N_\Lambda := \sum_{\mathbf{k}\in\Lambda^*} N_{\mathbf{k}}$. Hence, similarly to the kinetic-energy operator (17), the Hamiltonian of the HYL model for $a > 0$ gets the form (cf. [23, (1.2) and (1.3)])

$$H_\Lambda^{\mathrm{HYL}} := H_\Lambda^{\mathrm{MF}} + \frac{a}{2V}\{N_\Lambda^2 - \sum\nolimits_{k\in\Lambda^*} N_k^2\}, \tag{51}$$

$$H_\Lambda^{\mathrm{MF}} := \sum\nolimits_{k\in\Lambda^*} \varepsilon_k\, N_k + \frac{a}{2V} N_\Lambda^2, \tag{52}$$

where H_Λ^{MF} is the Hamiltonian of the model with the *mean-field* (MF) interaction.

We recall that stabilising MF interaction in (52) "improves" the properties of the ideal Bose gas in such a way that for chemical potential in this model $\mu \in (-\infty, +\infty)$ is allowed. However, the MF model keeps, due to the saturation mechanism, the *zero-mode* BEC intact with the *same* as for the IBG critical density $\rho_c(\beta) < \infty$ for $d > 2$ with the total amount of condensate density $\rho_0^{\mathrm{MF}}(\beta,\mu) = \rho(\beta,\mu) - \rho_c(\beta)$, for $\mu > \mu_c^{\mathrm{MF}}(\beta)$, see, e.g., [39, 40]. Here the particle mean density is $\rho(\beta,\mu) = \lim_{V\to\infty} \langle N_\Lambda\rangle_{H_\Lambda^{\mathrm{MF}}}(\beta,\mu)\,/V$. For the MF model, there is *no jump* of the condensate density at the BEC phase transition point $\mu_c^{\mathrm{MF}}(\beta)$; that is, one gets $\lim_{\mu\to\mu_c^{\mathrm{MF}}(\beta)+0} \rho_0^{\mathrm{MF}}(\beta,\mu) = 0$.

The difference in the behaviour of the condensate in models (51) and (52) reflects a difference in the origin (mechanism) of the BEC phase transition. In the mean-field model (52), similarly to ideal Bose gas, the condensation is a consequence of the balance between entropy and kinetic energy, which is the first term in the right-hand side of (52). The indicated term is minimal when all bosons occupy the *zero mode*: $\mathbf{k} = 0$, and this choice does *not* affect the interaction term.

On the other hand, the Huang–Yang–Luttinger model (51) is the MF-interacting Bose gas (52) (that manifests a *conventional* zero-mode BEC for $d > 2$) perturbed by the HYL-interaction term, the last item in (51). Then the effect, which favours the particle accumulation in zero mode by the kinetic energy term, is now *enhanced*, for *any* $d \geqslant 1$, by this last interaction-energy term since it has the smallest value when *all* bosons occupy the *same* energy mode. For that reason, the condensation in the HYL model is *non-conventional*. Indeed, it is a result of the balance between entropy and the interaction energy, which produces (non-conventional) zero-mode BEC even for $d \geqslant 1$.

This difference has a further consequence. In the *mean-field* model, the *conventional* condensation occurs if and only if it occurs in the corresponding *ideal* Bose gas: the mean-field critical density $\rho_c(\beta)$ is finite only for $d > 2$. While on the contrary, in the HYL model due to the particle interaction there is always the *zero-mode* BEC for sufficiently large density ρ and any $a > 0$. It occurs for any $d \geqslant 1$ even when for the non-perturbed MF-interacting boson gas the critical density $\rho_c(\beta)$ for $d \leqslant 2$ is *infinite* and then it does not manifest BEC.

2. JINR-Dubna, 1975: Experimental Observation of Bose–Einstein Condensation in Superfluid ^{4}He

"La seule vraie connaissance est la connaissance des faits."
Georges-Louis Buffon, *Histoire naturelle.*

2.1. Prehistory

The idea to scrutinise and to start experimental studies of the Bose–Einstein condensation (BEC) in superfluid ^{4}He (He II) was born at the Laboratory of Theoretical Physics (JINR) in 1973. It was formulated during a routine annual meeting of the Condensed Matter Research Group chaired by Professor V. G. Soloviev. Since our colleague V. B. Priezzhev had just defended his PhD thesis "Collective excitations in quasi-cristal models of liquids" (Dubna, 1973), where essential results concerned a quasi-cristal model of the liquid helium ^{4}He [41, 42], Professor Soloviev asked about applications of these results and suggested to contact our colleagues from the Laboratory of Neutron Physics. There they possessed a powerful pulsed reactor IBR-30 for *neutronography* (neutron scattering analysis) of liquid ^{4}He. This equipment could open an eventual possibility to confirm a long-standing hypothesis about existence of BEC in He II, which was a main hypothesis of the *Bogoliubov theory* of superfluidity [43–45].

So, armed with Priezzhev's thesis and my student papers [46, 47] about liquid ^{4}He, we contacted the Laboratory of Neutron Physics and one of the leading experts Zh. A. Kozlov, who had already been involved in neutron scattering experiments with liquid ^{4}He, cf. [48, 49]. The collaboration started with preparation of theoretical basis for the future experiments.

At that time, one of the theoretical observations interesting for our project was published by P. C. Hohenberg and P. M. Platzman [50]. They supposed that the *high-energy* neutrons (with a very short *de Broglie* wave-length) scattered off of single helium atoms may provide information about the *zero-momentum* BEC in *superfluid* ^{4}He (He II). After *deep-inelastic* scattering from individual "condensed" atoms, the energy transferred from the neutron would be *almost* equal to the recoil energy *broadened* by *final-state* interactions (negligible with respect to the large recoil energy). That is, after being struck by the high-energy neutrons, the single helium atoms recoil as if they were *free*. While for atoms *out* of BEC the high-energy transfer would be the recoil energy *broadened* by the *Doppler shifts* because of non-zero momenta of "non-condensed" atoms. The (optimistic) estimates given in [50] show that the Doppler broadening will be several times larger than the broadening expected because of the final-state interactions. Then neutron scattering cross sections are anticipated to show *two* components: a *narrow* one for scattering from the BEC and a *wider* one for scattering from "non-condensed" atoms of ^{4}He.

Before 1974 at least *three* significant experiments [51–53] had been carried out to check the neutron scattering suggestion formulated in [50]. The results of these papers were severely censured by H. W. Jackson [54] in a long accurate paper that scrutinised these experiments. In spite of the fact that in [51–53] the experimental data were interpreted as giving evidence for a condensate fraction ranging from 2.7 to 17%, the conclusion in [54] was that the data of these three experiments appear to be consistent with the *absence* of the condensate, or no more than a *fraction* of 1%. This pessimistic conclusion was in a certain conflict with some theoretical estimates of the condensate.

As we have seen in Subsubsections 1.2.2 and 1.2.3, the notion of the *condensate* of bosons has to be reexamined when we consider the interacting systems. Therefore, it is appropriate to clarify this notion, since the Bogoliubov theory [43–45] *insists* on both items: the *condensate* of bosons and the necessity of *interaction* to develop a *microscopic* explanation of superfluidity of He II. Yet, the *same* value of interaction may completely destroy condensate, intuitively presented as a fraction of particles, which does not move, "frozen in the momentum space" with $\mathbf{k} = 0$. This picture poses no problem for the perfect Bose gas, where the particle-number operator $N_{\mathbf{k}}$ in a given mode, $\mathbf{k} \in \Lambda^*$, is the integral of motion. Since single-particle quantum states are not proper for interacting system, L. Onsager and R. Penrose (1956) [55, 56], worked out a definition of *condensate* in this case. Their proposal was to identify condensation with an *Off-Diagonal Long-Range Order* (ODLRO), related to asymptotics of one-particle *density matrix*. This criterion shows that expectation (*mean value*) of the occupation number for *zero-mode* $\mathbf{k} = 0$ can still be used as a characterization of the Bose–Einstein condensation. Moreover, they have given about 8% as estimate of the fraction of particles density in liquid ^{4}He having $\mathbf{k} = 0$ (BEC) at $T = 0$ K.

This was a reason to *revise* in [57–59] (1973–1978) the conclusion of [54] by experiment along the Hohenberg–Platzman suggestions [50] using a new equipment available at that time in the Laboratory of Neutron Physics, JINR-Dubna. Namely, it was the IBR-30 *pulsed reactor* ensuring the *deep-inelastic* neutron scattering experiments on liquid ^{4}He together with the DIN-1M spectrometer, the time-spectrum analyser channel width being 8 μs.

2.2. Deep-inelastic neutron scattering and the Bose–Einstein condensate

The neutron–^{4}He-atoms inelastic *double-differential* scattering cross section for N atoms is given in the first *Born approximation* by formula involving the van Hove dynamic *structure factor* $S(k,\omega)$ [60]:

$$\frac{d^2\sigma}{d\Omega dE_f} = N\,\frac{m_n^2}{(2\pi)^3\hbar^5}\,\frac{k_f}{k_i}\,|\widehat{U}(k)|^2\,S(k,\omega). \tag{53}$$

One can find more details and explanations in the recent book [61, §4.1, §4.2]. Here for neutron with mass m_n, initial momentum $\hbar\,\mathbf{k}_i$ and energy $E_i = (\hbar k_i)^2/2m_n$, we denote by $\hbar\mathbf{k} = \hbar\mathbf{k}_i - \hbar\mathbf{k}_f$ and $\hbar\,\omega = E_i - E_f$ the *transferred* momentum and energy corresponding to their finite values $\hbar\mathbf{k}_f$ and $E_f = (\hbar k_f)^2/2m_n$. Function $\widehat{U}(k)$ is the Fourier transform of the neutron–helium atom interaction. In the range of large energies and momenta transfer considered in [57], one can use in (53) the *pseudo-potential* approximation with the corresponding scattering length a, see [58]. Then (53) can be rewritten as

$$\frac{1}{N}\,\frac{d^2\sigma}{d\Omega dE_f} = \frac{\sigma_b}{8\pi^2\hbar}\left(1-\frac{\hbar\omega}{E_i}\right)^{1/2} S(k,\omega), \quad E_f = E_i - \hbar\omega, \tag{54}$$

where $\sigma_b := (1 + m_n/M_{\mathrm{He}})^2\,4\pi a^2$ is the bound-helium-atom cross section.

Taking into account that $S(k,\omega)$ in (53) (or in (54)) is the Fourier transform of *density–density* correlation function [60], [61, §4.2], Hohenberg and Platzman [50] *argued* that for *high-energy* incident neutrons and *deep-inelastic* scattering (*high* transfer of energy and momentum) the van Hove factor gets the form (*Impulse Approximation*)

$$S^{\mathrm{IA}}_{\beta,\mu}(k,\omega) = \lim_{V\to\infty}\frac{1}{V}\sum_{\mathbf{q}\in\Lambda^*} n_q^{\mathrm{He}}(\beta,\mu)\;\delta\left(\hbar\omega - \frac{\hbar^2}{2M_{\mathrm{He}}}(\mathbf{k}^2 + 2\,\mathbf{k}\cdot\mathbf{q})\right), \tag{55}$$

where $n_q^{\mathrm{He}}(\beta,\mu) := \langle N_q\rangle_{H_\Lambda^{\mathrm{He}}}(\beta,\mu)$ are mean values of occupation numbers in modes $\mathbf{q} \in \Lambda^*$ for atoms of liquid ^{4}He, cf. (20) for the ideal Bose gas.

Recall that the high-energy neutron *deep-inelastic* scattering (also called the *neutron Compton scattering* [61, §5.4, 5.4.2]) is such that, during the short time interval t_S of a collision, the force of impact is much larger than all the other forces. Therefore, counting in this period (the *short-time scattering* approximation), we can consider the other forces to be negligible. Then a *noncoherent* scattering will be essentially defined by *momentum distribution* of individual targets (atoms of ^{4}He). This is expressed by formula (55). For that reason, the accuracy of the *Impulse Approximation* gets better for growing values of (k, ω) (that is, for shorter t_S) and gives a direct information about momenta of helium atoms. This would manifest itself as a *two-component* structure of the van Hove factor, which is the sum of a "narrow peak" corresponding to the scattering on a *condensate* and a "wide background" corresponding to the scattering on moving helium atoms *out* of the condensate.

Since values of momentum and energy and transfers $(k, \hbar\omega)$ are bounded, one has to take into account the impact of *finite-state* interaction of the helium atom with environment, that evidently breaks the kinematic relations in (55). As a consequence, formula (55) needs corrections, see discussion in [61, §5.4]. On that account, it is instructive to check the two-component *ansatz* by applying (55) for the *ideal* Bose gas, where there are no *finite-state* interactions at all. Therefore, we insert into formula (55) the ideal Bose-gas density distribution $n_q(\beta, \mu) = \left(e^{\beta\,(\varepsilon_q - \mu)} - 1\right)^{-1}$, see (20).

For a given density of bosons $\rho > 0$, on account of (55) and (25), one obtains for chemical potentials $\{\mu_\Lambda(\beta, \rho)\}_\Lambda$ that

$$S_{\beta,\mu_\Lambda(\beta,\rho)}(k, \omega) = \lim_{V\to\infty} \frac{1}{V} \sum_{\mathbf{q}\in\Lambda^*} n_q(\beta, \mu_\Lambda(\beta, \rho))\; \delta\left(\hbar\omega - \frac{\hbar^2}{2m}(\mathbf{k}^2 + 2\,\mathbf{k}\cdot\mathbf{q})\right). \tag{56}$$

Then for $\rho > \rho_c(\beta)$, that is, in the presence of BEC, by virtue of (23) and (27)–(29) for $\lim_{V\to\infty} \mu_\Lambda(\beta, \rho) = 0$, we deduce from (56) for *two* components of the van Hove dynamic structure form factor the following representation:

$$S_{\beta,\mu=0}(k, \omega) = \rho_0(\beta)\,\delta\left(\hbar\omega - \frac{\hbar^2}{2m}\mathbf{k}^2\right) + \frac{m}{(2\pi\,\hbar)^2 k} \int_{\Delta(k,\omega)}^{\infty} dq\, q\, \left(e^{\beta\,\varepsilon_q} - 1\right)^{-1}, \tag{57}$$
$$\Delta(k, \omega) := \frac{|\hbar\omega - \varepsilon_k|\, m}{\hbar^2 k}.$$

So, as we discussed above, for the ideal Bose gas without the *final-state* atom interactions the neutrons scattering from "condensed" atoms yields in the dynamic structure function $S_{\beta,\mu=0}(k, \omega)$ (57) the $\hbar\omega$-dependent *narrow* "δ-function" contribution supported on the *one-particle* spectrum: $\varepsilon_k = (\hbar\mathbf{k})^2/2m$, see the first component in the right-hand side of equation (57). Interactions of the recoiled atoms in the *final state* is the reason of broadening of this sharp peak.

The *wider* last term (the second component) in the right-hand side of equation (57) is a large peak as a function of transferred energy $\hbar\omega$ with maximum again at ε_k. It is a result of the *Doppler* broadening of the transferred energy owing to the neutron scattering on the moving "non-condensed" atoms.

2.3. Description of experiment and results (JINR, 1975)

2.3.1. Description of experiment

The experiment in [57, 58] was carried out on the IBR-30 pulsed reactor in booster regime, using a DIN-1M spectrometer. A monochromatic beam of neutrons with energy $E_i = 189.4$ meV was analysed, after scattering on the sample at angle $\theta = 122.62^\circ$, by the *time-of-flight* method between the sample and the detector. Values of the transferred momentum that were most favorable for the experiment were chosen in the range $k \sim (13-15)$ Å^{-1}. The lower bound is determined (for *deep-inelastic scattering*) by the closeness of the approach to *unity* of the ratio of the energy transferred in the neutron scattering to the energy of the free helium atom that corresponds to the momentum $\hbar k$: $\varepsilon_{\mathrm{He}}(k) = (\hbar k)^2/2M_{\mathrm{He}}$. This ratio was of the order of 0.96 at $k = 14.1$ Å^{-1} in our experiment [48]. This interval is limited above by the resolving power of the spectrometer. In this experiment, the width of the resolution function in the region of the "helium" peak was equal to the value ~ 9 meV.

We did not strive for the limiting parameters of the resolution function, since the condensate part of the "helium" peak is broadened by the interaction in the *final state* to the extent that it is not possible to separate it in explicit form. Therefore, attention was concentrated on lowering the statistical error and obtaining the highest possible accuracy in measurement of the shape of the "helium" peak.

Over the time of the neutron-scattering experiment at 1.2 K, the integrated count in the "helium" peak amounted to $\sim 2 \cdot 10^5$ pulses. The experimental spectra of inelastic neutron scattering by liquid helium at temperatures of 1.2 and 4.2 K were measured. The width of the time spectrum analyser channel was 8 μs. The total time of measurement amounted to 240 hours. The measurements at the different temperatures were not normalised. The energy resolution was determined with the help of a *vanadium* sample and converted for the inelastic-scattering region.

The *background* was measured during evacuation of the helium vapor over the liquid at the bottom of the cryostat. The center of the elastic peak is located in the 172nd channel. The relatively large scatter in the results at the wings of the "helium" peak is explained by the fact that the background was measured over times *less* than the effect and was not smoothed, but was calculated from the channel. The experimental results were corrected for the effectiveness of the detector.

2.3.2. Observation of Bose–Einstein condensate in superfluid He II

A numerical analysis of the experimental data was carried out on the basis of the regularised iterative process of Gauss–Newton method [62, 63] (library program CÔMPIL, C-401, Dubna).

As a result of this analysis, it has been established that the model with *two Gaussian* components (cf. (57)) for the double-differential cross section of deep-inelastic high-energy neutron scattering by ^{4}He describes the experimental data sufficiently well. It was also established that complication of the two-component approximating model by the addition of more *trial* Gaussians does not improve it.

Further, by comparison of quantities of the χ^2-criterion for the two temperatures (below and above the λ-point: $T = 2.17$ K), we verify that the model with *two* Gaussians is *better* from the viewpoint of the statistical criteria at $T_1 = 1.2$ K, whereas the model with *one* Gaussian is *better* at $T_2 = 4.2$ K, cf. Figure 1.

Finally, based on this double-Gaussian observation, we obtain the estimate $(3.6 \pm 1.4)\%$ for the value of the *fraction* ρ_0/ρ for the *Bose–Einstein condensate* ρ_0 at $T_1 = 1.2$ K. Our analysis also provided an evidence that to *lower* the impact of the interaction of the helium atoms in the *final state* (thus to improve a reliability of the estimate of the fraction ρ_0/ρ), the Bose condensate must be studied at *higher* energies of the incident neutrons together with improvement of the apparatus *resolution function*.

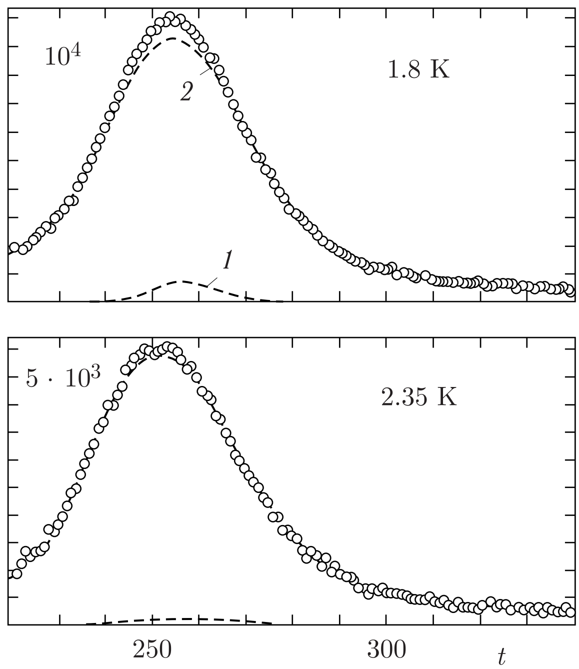

Figure 1. Experimental spectrum of scattered neutrons (measured in liquid ^{4}He throughout the structure factor $S(k, \omega)$) for $k = 13.4$ Å^{-1}. The ordinate is the number of counts (pulses) per unit solid angle. The upper part of the figure corresponds to the temperature $T = 1.8$ K $< T_\lambda$. The dotted line, which indicates *peak 1* at the bottom, corresponds to recognised fraction of the condensate in this experiment. The dotted line *1* together with dotted line *2* represents the method of *two-Gaussian* resolution of the spectra of neutrons scattered by liquid ^{4}He at temperature $T = 1.8$ K. The non-dashed curve matches with the sum of lines *1* and *2*. The lower part of the figure corresponds to the temperature $T = 2.35$ K $> T_\lambda$. Then fitting the data with a single Gaussian function provides a good match to the experimental spectrum curve.

For further discussion of the last subtle point concerning a balance between *Doppler broadening* and the *width* of the resolution function for *increasing* energy of the incident neutrons, we refer to a very complete review [49, Ch. 3, Secs. 3.1–3.4]. For illustration see Figure 1. It is taken from [49], cf. Figure 25.

2.4. Observation of temperature dependence of Bose–Einstein condensate in liquid ^{4}*He*

At that time the Laboratory of Neutron Physics of JINR also became a pioneer in observation of *temperature dependence* of the Bose–Einstein condensate in liquid ^{4}He [59]. Besides, in this way one can revise the long-time *open question* about emergence of the condensate in liquid ^{4}He together (*simultaneously*!) with the superfluidity.

Experiment in [59] was carried out using the *time-of-flight* method with a DIN-1M spectrometer in the booster regime of the IBR-30 reactor, which is similar to that in Subsubsection 2.3.1. The spectra of the neutrons scattered by liquid ^{4}He were measured simultaneously at three scattering angles $\theta = 122.6°$, $109.5°$ and $96.5°$ for initial neutron energy $E_i = 190$ meV. The aim was to analyse the *temperature dependence* of the relative density of the Bose–Einstein condensate in liquid ^{4}He by studying the spectra of deep-inelastic neutron scattering at momentum transfers corresponding to $k = 12{-}14$ Å^{-1} and temperatures $T = 1.2{-}4.2$ K.

Note that a cryostat with ^{4}He vapor pumped on was used in the measurement. The helium temperature was determined by measuring the vapor pressure over the liquid and with the aid of a carbon resistor. The temperatures in the intervals 1.2–1.8 K and 1.95–4.2 K were maintained with accuracy to ~ 0.02 K and ~ 0.01 K, respectively. The backgrounds of the empty cryostat and of the cryostat filled with liquid helium were determined by performing the measurements alternately using a cadmium shutter to block the neutron beam and without the shutter.

The main results of the experiment [59] are the following (below we include some precisions due to later experimental data from [49]):

1. The relative density ρ_0/ρ of the Bose–Einstein condensate was calculated similarly to [57, 58], by the method of *two-Gaussian* resolution of the spectra of neutrons scattered by liquid ^{4}He in the temperature interval $T = 1.2{-}1.8$ K.

2. The Bose–Einstein condensate was observed for $T < T_0$, whereas for $T \geqslant T_0$, within the limits of the accuracy of the experiment and of the mathematical reduction, *no* Bose–Einstein condensate was observed, cf. Figure 2. The relative density ρ_0/ρ of the Bose–Einstein condensate was estimated using the formula

$$\frac{\rho_0}{\rho} = \frac{S_{\rm BC}}{S_{\rm BC} + S_{\rm SC}}. \tag{58}$$

Here in the framework of the *two-Gaussian* resolution the value of $S_{\rm BC}$ is the *area* below the spectrum of neutrons scattered on the Bose–Einstein condensate, thought $S_{\rm SC}$ (in the two-Gaussian resolution) is the *area* below the spectrum, which is due to the scattering on non-condensed atoms of liquid ^{4}He, see Figure 1. Then the temperature dependence of the relative density fits into the form [59]

$$\frac{\rho_0}{\rho} = \xi_0 \left(1 - \left(\frac{T}{T_0}\right)^m\right), \quad T \leqslant T_0 = (2.22 \pm 0.05)\ \text{K}, \tag{59}$$

where

$$\xi_0 = (7 \pm 0.5)\% \quad \text{and} \quad m = 9 \pm 4. \tag{60}$$

The temperature-dependent experimental data for $\xi = \rho_0/\rho$ (small *black* squares) are presented in Figure 2, taken from the review [49] (cf. Figure 27).
The analysis is essentially concentrated in the *vicinity* of the point T_0 and appeals for more accuracy.

3. The Bose-condensation temperature T_0 (59) within the limits of the *accuracy* of the experiment is very close to the temperature $T_\lambda = 2.17$ K (λ-point) of transition of liquid ^{4}He into He II, that is, into *superfluid* phase for decreasing temperature.

4. Character of the temperature dependence of the Bose–Einstein condensate relative density (59) is *similar* to the temperature dependence of the relative density

$$\frac{\rho_s}{\rho} = 1 - \left(\frac{T}{T_\lambda}\right)^{5,6}, \quad T \leqslant T_\lambda = 2.17\ \text{K}, \tag{61}$$

for *superfluid* component ρ_s, see [64, 65], [66, Ch. 13, Sec. 13.2], whereas for the ideal Bose gas index $m = 3/2$ [6, Ch. III, §2].

In conclusion, we mention a radically different method [67] for studying the Bose condensation in liquid ^{4}He, see also [49, 59]. One of the possible ways for measuring the Bose–Einstein condensate density and investigating the connection between *condensation* and *superfluidity*

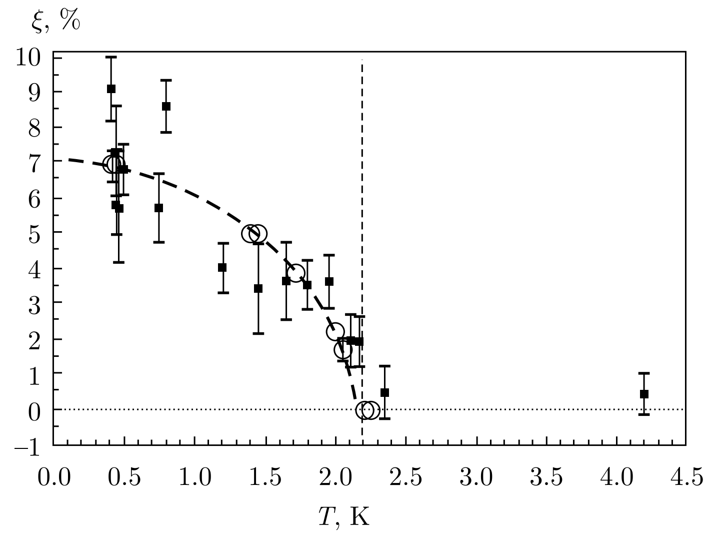

Figure 2. Experimental data (small *black* squares) for relative density $\xi = \rho_0/\rho$ (59) of the Bose–Einstein condensate in the temperature interval $T = 0.4{-}4.2$ K. Amount of condensate fraction at each temperature was calculated by the method of *two-Gaussian* resolution for spectrum of deep-inelastic neutron scattering by liquid ^{4}He, cf. (58) and Figure 1.

consists in performing experiments with deep-inelastic scattering of neutrons on liquid ^{4}He with a small *admixture* of ^{3}He.

The presence of ^{3}He with concentration $c > 0$ shifts the *superfluid transition* (along the λ-line on the phase-diagram of the ^{3}He–^{4}He mixture) to lower temperatures $T_\lambda(c > 0) < T_\lambda$. For that reason, ^{3}He impurities are able to destroy superfluidity and thus the Bose–Einstein condensate, see [68, 69]. This observation and similarity between (59) and (61) would allow one to establish a *direct* correlation between relative densities of Bose condensation and of superfluidity in He II.

To this end a description of possible experiment has been presented in [67] for $c = 5\%$. Then $T_\lambda(5\%) = 2$ K and by using (59) one obtains that for *pure* system at this temperature $(\rho_0/\rho)(c = 0,\ T = 2\text{ K}) \simeq 2\%$. If there is a *correlation* between superfluidity and Bose condensation, then since for $c = 5\%$ one obtains $(\rho_s/\rho)(c = 5\%,\ T = 2\text{ K}) = 0$, one *should* also observe the same value, $(\rho_0/\rho)(c = 5\%,\ T = 2\text{ K}) = 0$, in the deep-inelastic neutron scattering experiments.

Calculations in [67] showed that in view of the *large* cross section for neutron *capture* by ^{3}He, to maintain the accuracy of observations, the neutron flux density in this experiment must be *larger* by approximately two orders of magnitude than the existing flux. Another problem related to this phenomenon is a supplementary *heating* because of the reaction

$$n + {}^3\text{He} \to {}^4\text{He} + (\sim 0.766\text{ meV}), \tag{62}$$

as well as a (tiny) gradual *decreasing* of the concentration c. The estimates in [67] revealed that the problem of heating is solvable, but the first problem needs a new source of neutrons and new spectrometer.

For relatively recent results on the properties of He II, including some experimental data about the Bose–Einstein condensate in liquid ^{4}He, see (a quite biased) review article [70].

3. Condensation and Bogoliubov Theory of Superfluidity

"To construct a complete molecular theory of superfluidity, it is necessary to consider the liquid helium as being a system of interacting atoms."
Nikolai Nikolaevitch Bogoliubov, *Lectures on Quantum Statistics.*

"Faced with this failure, theorists retreated into the corner of low density gases with weak interaction."
Elliott H. Lieb, *The Bose Fluid.*

After a guess about correlation between superfluidity and Bose–Einstein condensate in liquid ^{4}He formulated by London in 1938 [2, §3], it was N. N. Bogoliubov who proposed in 1947 [43–45] an elegant theory that *links* the Bose–Einstein condensation with superfluidity. As a matter of fact, London was trying to develop a "highly idealised model" for superfluidity of liquid ^{4}He by taking into account only the ideal Bose gas with condensate and the quadratic *one-particle* spectrum: $\varepsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/(2m)$, see [2, §3]. However, according to the arguments based on the *Landau criterion* of superfluidity [71, 72], the ideal Bose gas with condensate cannot manifest the superfluidity (zero viscosity) because of these *low-lying* quadratic one-particle excitations, cf. [71].

In [43–45] Bogoliubov declared that for superfluidity it is necessary to consider the liquid helium as being a system with *interaction* and also with Bose–Einstein *condensate* favouring *collective* (instead of *one-particle*) excitations of the condensed "helium jelly". On that account, Bogoliubov selected as an essential *ansatz* the interaction between *condensate* and *out-of-condensate* atoms, which is such that it is *weak* enough to preserve condensate in the zero-mode $\mathbf{k} = 0$. This *off-diagonal* interaction in the Bogoliubov Weakly Imperfect Bose Gas (WIBG) [73] yields necessary modification of the low-lying spectrum and produces the *collective* Bogoliubov excitations. Since they satisfy the *Landau criterion* [71], this *ansatz* completes the Bogoliubov theory of superfluidity [43, 44].

It is worth mentioning here that particle interaction may (in turn) modify the *nature* of the Bose–Einstein condensate to a *generalised* or *non-conventional* one. To this end, see Subsubsections 1.2.2 and 1.2.3. This, for example, would have impact on the properties of the WIBG. For further details about condensate in the Bogoliubov model of WIBG and superfluidity, see [25, Sections 5 and 6].

P.S.

I would like to note that an excellent review article [74] covers a *point*, which is deliberately missed in the present *double-jubilee* message dedicated *uniquely* to formulation of the Bose–Einstein condensation concept in 1925 and to observation of the Bose–Einstein condensation in superfluid ^{4}He (He II) at JINR-Dubna, which was published in 1975. This non-mentioned *point* involves enormous literature about a special kind of condensation of dilute *ultracold* atomic Bose gases in traps, which inherits, see, for example, [30], some ideas of Subsubsection 1.2.2 related to different *types* of condensation. Curious readers will find a lot of information about this missed *point* in paper [74].

Another important missing *point* is mentioned in the *epigraph* by Immanuel Kant, which is quoted before Section 1. It concerns the mathematical *status* of the concept of Bose–Einstein condensation. For the mathematically-minded readers it would be instructive to consult first an excellent, short and comprehensive survey [75] about a variety of mathematically rigorous results, also including those on the *ultracold* atomic Bose gases in traps. Besides, there is a variety of mathematical results, for example, [76], or those collected in the book [27], which study the Bose–Einstein condensation in the framework of abstract mathematical approach to Quantum Statistical Mechanics [77].

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Conflicts of interest

The author declares no conflicts of interest.

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