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New variants of $\mathcal{N} = 3, 4$ superconformal mechanics

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Abstract

We construct superconformal mechanics with $\mathcal{N} = 3$ and $\mathcal{N} = 4$ supersymmetries that were inspired by analogy with the supersymmetric Schwarzian mechanics. The Schwarzian, being another system with superconformal symmetry, provides insight into the field content of supersymmetric mechanics, most notably, on the number and properties of the fermionic fields involved. Adding more fermionic fields (four in the $\mathcal{N} = 3$ case and eight in the $\mathcal{N} = 4$ case) made it possible to construct systems possessing maximal superconformal symmetries in $\mathcal{N} = 3$ and $\mathcal{N} = 4$, namely $osp(3|2)$ and $D(1,2;\alpha)$. In the case of $\mathcal{N} = 4$ supersymmetry, we explicitly construct a new variant of $\mathcal{N} = 4$ superconformal mechanics in which all bosonic subalgebras of the $D(1,2;\alpha)$ superalgebra have a bosonic realization. In addition, the constructed systems involve the $so(3)$ currents whose parameterization is not fixed, which allows one to consider different underlying geometries.

Keywords: superconformal mechanics, $\mathcal{N}$-extended supersymmetry, integrable system, Schwarzians

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1. Introduction

Recently, new interest in superconformal mechanics has arisen due to the intense search for AdS_2 solutions of both Type II and eleven-dimensional supergravities. Several recent works have classified possible ten- and eleven-dimensional AdS_2 solutions with different numbers of preserved supersymmetries [1–4]. These solutions are especially interesting due to the high dimensionality of the internal space, which offers many possibilities for realizing supersymmetry. The presence of AdS_2 factors leads to dual superconformal quantum mechanics with different numbers of supersymmetries. These dual superconformal quantum mechanics should be the basis for a microscopical description of black holes with the geometries near the horizon. Unfortunately, until now, mainly superconformal mechanics with $\mathcal{N} = 4$ and/or $\mathcal{N} = 8$ supersymmetries have been fully analyzed.

For example, even the simplest case of $\mathcal{N} = 3$ superconformal mechanics, to our knowledge, has not been studied so far in the literature. This forces us to construct and analyze

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$\mathcal{N}=3$ superconformal mechanics (Section 2). Our analysis shows that one needs to deal with four fermionic components to describe such a system. Moreover, under such circumstances, there is no possibility to avoid extending $OSp(3|2)$ supersymmetry to the $OSp(4|2)$ one, thus constructing a system with $\mathcal{N}=4$ supersymmetry.

It should be noted that two variants of $\mathcal{N}=3, d=1$ superconformal systems were constructed in the papers [5, 6] within the so-called supersymmetric Schwarzian mechanics. However, a direct relation between superconformal and super-Schwarzian mechanics was not established. In Section 3, we provide a detailed consideration of the $\mathcal{N}=3$ Schwarzian mechanics and demonstrate its equivalence to the $\mathcal{N}=3$ mechanics considered in Section 2. The most interesting result of this section is the irreducibility constraints on four $\mathcal{N}=3, d=1$ superfields leading to the supermultiplet $(4,4,0)$.

Despite the fact that $\mathcal{N}=4$ superconformal mechanics with $D(1,2;\alpha)$ dynamical symmetry were widely explored in the literature [7–23], in all the cases, only one $su(2)$ subalgebra from the $sl_2\times su(2)\times su(2)$ bosonic core of the $D(1,2;\alpha)$ superalgebra has a bosonic realization. The second $su(2)$ subalgebra has only a fermionic realization. However, within the AdS_2 solutions of ten-dimensional supergravity with bosonic $sl_2\times S^3\times S^3$ metrics, both S^3 spheres have a bosonic realization [1–4]. In Section 4, we explicitly construct a possible variant of dual superconformal mechanics in which all bosonic subalgebras of the $D(1,2;\alpha)$ superalgebra have a bosonic realization. Finally, we conclude with a short review of the obtained results and with possible future developments.

2. $\mathcal{N}=3$ superconformal mechanics

2.1. Basic ingredients

To construct $\mathcal{N}=3$ superconformal mechanics with $OSp(3|2)$ dynamical supersymmetry, let us introduce the dilaton r, the corresponding momentum p_r, and three currents J_i, which obey the following nonzero Poisson brackets:

$$\{p_r,\, r\}=1,\quad \{J_i,\, J_j\}=\epsilon_{ijk}\,J_k. \tag{2.1}$$

The fields r, p_r will be used to construct the $sl(2,\mathbb{R})$ part of the $osp(3|2)$ superalgebra with the generators H, D, K defined as

$$H=\frac{1}{2}\,p_r^2+\frac{\mathcal{A}}{r^2},\quad D=\frac{1}{2}\,r\,p_r,\quad K=\frac{r^2}{2},$$
$$\{D,H\}=-H,\quad \{D,K\}=K,\quad \{H,K\}=2D. \tag{2.2}$$

Here, the angular part of the Hamiltonian $\mathcal{A}$ depends only on the bosonic $so(3)$ currents J_i (2.1) and fermions, which will be defined shortly [10].

To construct three supercharges Q_i entering into the $osp(3|2)$ superalgebra, one has to introduce, as we noted in the Introduction, four fermions, namely three fermions ψ_i and an additional single fermion χ, which obey the following nonzero Poisson brackets:

$$\{\psi_i,\psi_j\}=\mathrm{i}\,\delta_{ij},\quad \{\chi,\chi\}=\mathrm{i}. \tag{2.3}$$

The superconformal charges of the $osp(3|2)$ superalgebra are realized with the help of the dilaton r and fermions ψ_i as

$$S_i=r\,\psi_i,\quad \{S_i,S_j\}=2\mathrm{i}\delta_{ij}\,K. \tag{2.4}$$

Using four fermions χ, ψ_i, one can construct the $so(4)$ currents

$$\widehat{J}_i = \frac{\mathrm{i}}{2}\,\epsilon_{ijk}\psi_j\,\psi_k, \quad \widehat{W}_i = \mathrm{i}\,\psi_i\,\chi, \tag{2.5}$$

which obey the standard $so(4)$ Poisson brackets

$$\left\{\widehat{J}_i,\,\widehat{J}_j\right\} = \epsilon_{ijk}\,\widehat{J}_k, \quad \left\{\widehat{W}_i,\,\widehat{J}_j\right\} = \epsilon_{ijk}\,\widehat{W}_k, \quad \left\{\widehat{W}_i,\,\widehat{W}_j\right\} = \epsilon_{ijk}\,\widehat{J}_k. \tag{2.6}$$

2.2. Supercharges and Hamiltonian

From the previous section, we learned that in our construction of superconformal $osp(3|2)$ dynamical supersymmetry, the generators of the dilatation D, conformal boost K, conformal supercharges S, which form the conformal part of the superalgebra, and the generators of $so(3)$ R-symmetry are already defined in (2.2), (2.4), (2.1), (2.5). Moreover, the structure of the supercharges Q_i is partially fixed to be

$$Q_i = p_r\,\psi_i + \frac{1}{r}\left[(R\text{-symmetry generators}) \times \text{fermions}\right]. \tag{2.7}$$

This structure of the supercharges has been advocated in [26] and then successfully applied to $\mathcal{N}=8$ superconformal mechanics in [27, 28].

Using the Anzatz (2.7), the supercharges of $\mathcal{N}=3$ superconformal mechanics can be easily found to be

$$Q_i = p_r\,\psi_i + \frac{1}{r}\left(\epsilon_{ijk}\,J_j\,\psi_k + J_i\,\chi\right) \quad\rightarrow\quad \{Q_i, Q_j\} = 2\,\mathrm{i}\,\delta_{ij}H, \tag{2.8}$$

where the Hamiltonian H reads

$$H = \frac{1}{2}p_r^2 + \frac{1}{2r^2}J_i\left(J_i - 2\widehat{J}_i + 2\widehat{W}_i\right). \tag{2.9}$$

Note that supercharges (2.8) satisfy supersymmetry algebra without placing any algebraic constraints on the generators J_i. It is important that without the bosonic current J_i one can construct only free supercharges $Q_i = p_r\,\psi_i$ and a free purely bosonic Hamiltonian $H = \frac{1}{2}\,p_r^2$.

To visualize the dynamical symmetry of the system, one has to calculate the Poisson brackets between the Poincaré (2.8) and conformal supersymmetry generators S_i (2.4):

$$\{S_i, Q_j\} = 2\mathrm{i}\,\delta_{ij}D + \mathrm{i}\,\epsilon_{ijk}\left(J_k + \widehat{J}_k\right). \tag{2.10}$$

Having in mind the brackets

$$\left\{J_i + \widehat{J}_i, Q_j\right\} = \epsilon_{ijk}Q_k, \quad \left\{J_i + \widehat{J}_i, J_j + \widehat{J}_j\right\} = \epsilon_{ijk}\left(J_k + \widehat{J}_k\right), \tag{2.11}$$

we conclude that the generators $\{Q_i\ (2.8), H\ (2.9), S_i\ (2.4), D, K\ (2.2)\}$ and $\left\{J_i + \widehat{J}_i\ (2.1), (2.5)\right\}$ form the superalgebra $osp(3|2)$.

To complete this section, let us make several comments:

1. The unavoidable presence of four fermions in the game raises the question of existence of the fourth supercharge extending the dynamical supersymmetry $osp(3|2)$ to the $osp(4|2)$ one. Indeed, one can immediately suggest that an additional superconformal charge is

$$s = r\,\chi, \quad \{s, s\} = 2\mathrm{i}K. \tag{2.12}$$

Then $q = \{H, s\}$ appears to be the fourth supercharge

$$q = p_r\,\chi - \frac{1}{r}J_i\psi_i \quad\rightarrow\quad \{q, q\} = 2\,\mathrm{i}H, \quad \{q, Q_i\} = 0. \tag{2.13}$$

2. The $osp(4|2)$ superalgebra contains an additional superconformal generator of R-symmetry $\widehat{W}_i$ (2.5) which appears in the brackets of the generators Q_i and q with s:

$$\{s, Q_i\} = -\{S_i, q\} = \mathrm{i}\left(J_i - \widehat{W}_i\right), \quad \{s, q\} = 2\,\mathrm{i}\,D,$$
$$\left\{J_i + \widehat{J}_i, J_i - \widehat{W}_i\right\} = \epsilon_{ijk}\left(J_k - \widehat{W}_k\right), \quad \left\{J_i - \widehat{W}_i, J_j - \widehat{W}_j\right\} = \epsilon_{ijk}\left(J_k + \widehat{J}_k\right). \quad (2.14)$$

   Thus, the generators $\{Q_i, q, S_i, s, H, D, K, J_i + \widehat{J}_i, J_i - \widehat{W}_i\}$ span the $osp(4|2)$ superalgebra and the system possesses not just $\mathcal{N} = 3$ but $\mathcal{N} = 4$ supersymmetry. Note that the generators of the $so(3)$ subalgebra and of the $so(4)/so(3)$ coset have the same bosonic core but differ in the fermions.

3. It is natural to expect the possibility of constructing a full $D(1, 2; \alpha)$ superalgebra with given fields. The corresponding supercharges read

$$Q_i = p_r\,\psi_i - \frac{1}{r}\left[2\alpha\epsilon_{ijk}J_j\psi_k + 2\alpha J_i\,\chi + (1 + 2\alpha)\widehat{J}_i\chi\right],$$
$$q = p_r\chi + \frac{1}{r}\left[2\alpha J_i\psi_i + \frac{1 + 2\alpha}{3}\widehat{J}_i\psi_i\right], \quad \{Q_i, Q_j\} = 2\,\mathrm{i}\,\delta_{ij}H, \quad \{q, q\} = 2\,\mathrm{i}\,H \quad (2.15)$$

   with the Hamiltonian

$$H = \frac{1}{2}p_r^2 + \frac{1}{r^2}\left[2\alpha^2 J_iJ_i + 2\alpha J_i\widehat{J}_i - 2\alpha J_i\widehat{W}_i - \frac{1 + 2\alpha}{3}\widehat{J}_i\widehat{W}_i\right]. \quad (2.16)$$

   Note that the realization of the generator J_i in (2.15), (2.16) is arbitrary, provided it has zero brackets with other fields involved. In particular, one can consider its realization via new bosonic fields only, new fermionic fields or combinations of both, reproducing systems constructed in [9, 17, 18, 20].

4. Interestingly, our $\mathcal{N} = 3$ superconformal mechanics coincides with the $\mathcal{N} = 3$ supersymmetric Schwarzian mechanics [6] at the superfield level (see the next section).

3. $\mathcal{N} = 3$ supersymmetric Schwarzian mechanics

It is important to note that using the method of nonlinear realizations, it is possible to construct superconformal mechanics which includes interactions with non-Abelian currents. The basic steps are very similar to the construction of the supersymmetric Schwarzian mechanics [5, 6].

Let us consider the $\mathcal{N} = 3$ case. The starting point here is the element of the $OSp(3|2)$ supergroup:

$$g = e^{\mathrm{i}t\mathcal{P}}\, e^{\theta_i\mathcal{Q}_i}\, e^{\lambda_j\mathcal{S}_j} e^{\mathrm{i}z\mathcal{K}} e^{iu\mathcal{D}} e^{\mathrm{i}\phi_i\mathcal{J}_i}. \quad (3.1)$$

The generators here obey the relations [24, 25]:

$$\mathrm{i}\,[\mathcal{D}, \mathcal{P}] = \mathcal{P}, \quad \mathrm{i}\,[\mathcal{D}, \mathcal{K}] = -\mathcal{K}, \quad \mathrm{i}\,[\mathcal{K}, \mathcal{P}] = 2\mathcal{D},$$
$$\{\mathcal{Q}_i, \mathcal{Q}_j\} = 2\delta_{ij}\mathcal{P}, \quad \{\mathcal{S}_i, \mathcal{S}_j\} = 2\delta_{ij}\mathcal{K}, \quad \{\mathcal{Q}_i, \mathcal{S}_j\} = -2\delta_{ij}\mathcal{D} - \epsilon_{ijk}\mathcal{J}_k,$$
$$\mathrm{i}\,[\mathcal{D}, \mathcal{Q}_i] = \frac{1}{2}\mathcal{Q}_i, \ \mathrm{i}\,[\mathcal{D}, \mathcal{S}_i] = -\frac{1}{2}\mathcal{S}_i, \quad \mathrm{i}\,[\mathcal{K}, \mathcal{Q}_i] = -\mathcal{S}_i, \quad \mathrm{i}\,[\mathcal{P}, \mathcal{S}_i] = \mathcal{Q}_i,$$
$$\mathrm{i}\,[\mathcal{J}_i, \mathcal{Q}_j] = \epsilon_{ijk}\mathcal{Q}_k, \quad \mathrm{i}\,[\mathcal{J}_i, \mathcal{S}_j] = \epsilon_{ijk}\mathcal{S}_k, \quad \mathrm{i}\,[\mathcal{J}_i, \mathcal{J}_j] = \epsilon_{ijk}\mathcal{J}_k. \quad (3.2)$$

Here, t, θ_i are the coordinates of the $\mathcal{N}=3$, $d=1$ superspace, and u, z, ϕ_i, λ_i are supposed to be superfields on this space.

The Cartan forms invariant with respect to left multiplication are defined as

$$\Omega = g^{-1}dg = \mathrm{i}\omega_P \mathcal{P} + \left(\omega_Q\right)_i \mathcal{Q}_i + \mathrm{i}\omega_D \mathcal{D} + \mathrm{i}\left(\omega_J\right)_i \mathcal{J}_i + \left(\omega_S\right)_i \mathcal{S}_i + \mathrm{i}\omega_K \mathcal{K} \tag{3.3}$$

and explicitly read

$$\begin{aligned} \omega_P &= e^{-u}\left(dt + \mathrm{i}\, d\theta_i\, \theta_i\right) \equiv e^{-u}\triangle t, \\ \omega_D &= du - 2z\triangle t - 2\mathrm{i}\, d\theta_i \lambda_i, \\ \omega_K &= e^{u}\left(dz + z^2\triangle t + \mathrm{i}\, d\lambda_i\, \lambda_i + 2\mathrm{i}\, z d\theta_i\, \lambda_i\right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} \left(\omega_Q\right)_i &= \left(\hat\omega_Q\right)_j M_{ij}, \quad \left(\omega_S\right)_i = \left(\hat\omega_S\right)_j M_{ij}, \quad \left(\omega_J\right)_i = \left(\hat\omega_J\right)_j M_{ij} + \frac{1}{2}\epsilon_{ijk} dM_{jm} M_{km}, \\ M_{ij} &= \left(e^{\mathfrak{M}}\right)_{ij}, \quad \mathfrak{M}_{ij} = \epsilon_{ijk}\phi_k, \end{aligned} \tag{3.5}$$

where the hatted forms are

$$\begin{aligned} \left(\hat\omega_Q\right)_i &= e^{-\frac{u}{2}}\left(d\theta_i + \triangle t \lambda_i\right), \\ \left(\hat\omega_J\right)_i &= -\mathrm{i}\epsilon_{ijk}\left(d\theta_j \lambda_k + \frac{1}{2}\triangle t\; \lambda_j \lambda_k\right), \\ \left(\hat\omega_S\right)_i &= e^{\frac{u}{2}}\left(d\lambda_i - \mathrm{i}\lambda_i\lambda_j d\theta_j + z\left(d\theta_i + \triangle t\, \lambda_i\right)\right). \end{aligned} \tag{3.6}$$

Note that by construction M_{ij} is an orthogonal matrix, $\left(M^{-1}\right)_{ij} = M_{ji}$, $\det M = 1$.

The covariant derivatives on the (t, θ_i) superspace can be defined in the standard way as

$$\partial_t = \frac{\partial}{\partial t}, \quad D_i = \frac{\partial}{\partial \theta_i} - \mathrm{i}\theta_i \frac{\partial}{\partial t}, \qquad \{D_i, D_j\} = -2i\delta_{ij}\partial_t. \tag{3.7}$$

Superconformal mechanics usually contains much fewer components that are present in the θ expansions of the superfields u, z, ϕ_i, λ_i, and one should impose additional superconformally invariant irreducibility conditions. Let us take

$$\omega_D = 0, \quad \left(\omega_J\right)_i = \omega_P\, B_i + \left(\omega_Q\right)_i \Sigma. \tag{3.8}$$

Both of these conditions were encountered in the construction of the Schwarzian mechanics [6]. A nontrivial point in the second condition is the restriction $\Sigma_{ij} = \delta_{ij}\Sigma$ in the most general expansion of $\left(\omega_J\right)_i = \omega_P\, B_i + \left(\omega_Q\right)_j \Sigma_{ij}$. Note that the field Σ holds the place of the $\mathcal{N}=3$ Schwarzian.

The condition $\omega_D = 0$ expresses the superfields z and λ_i in terms of u:

$$z = \frac{1}{2}\dot u, \quad \lambda_i = -\frac{\mathrm{i}}{2} D_i u. \tag{3.9}$$

The second condition in (3.8) reads

$$\frac{1}{2}\epsilon_{ijk} D_m M_{jn}\, M_{lm}\, M_{kn} - \mathrm{i} M_{ij}\, M_{lm}\, \epsilon_{jmn}\lambda_n = \delta_{li} e^{-\frac{u}{2}}\Sigma = \delta_{li}\widetilde{\Sigma}. \tag{3.10}$$

To analyze the relation (3.10), it is useful to introduce some parameterization for the orthogonal matrix $M_{ij} = \left(e^{\mathfrak{M}}\right)_{ij}$, $\mathfrak{M}_{ij} = \epsilon_{ijk}\phi_k$. The most convenient one is just the stereographic projection

$$M_{ij} = \frac{1 - \frac{1}{4}y^2}{1 + \frac{1}{4}y^2}\delta_{ij} + \frac{\epsilon_{ijk} y_k}{1 + \frac{1}{4}y^2} + \frac{1}{2}\frac{y_i y_j}{1 + \frac{1}{4}y^2}, \quad y_i = \phi_i \frac{\tan\left(\frac{1}{2}\sqrt{\phi^2}\right)}{\frac{1}{2}\sqrt{\phi^2}}, \quad y^2 = y_i y_i, \quad \phi^2 = \phi_i\phi_i. \tag{3.11}$$

Substituting (3.11) and taking into account (3.9), one can reduce (3.10) just to

$$\frac{D_j y_i + \frac{1}{2}\epsilon_{imn} y_m D_j y_n}{1+\frac{1}{4}y^2} = \delta_{ij}\widetilde{\Sigma} + \frac{1}{2}\epsilon_{ijk} D_k u$$

or, equivalently,

$$D_j y_k = N_{km}\Lambda_{mj}, \quad \Lambda_{mj} = \delta_{mj}\widetilde{\Sigma} + \frac{1}{2}\epsilon_{mjn} D_n u, \quad N_{ij} = \delta_{ij} + \frac{1}{2}\epsilon_{ijk} y_k + \frac{1}{4} y_i y_j. \tag{3.12}$$

Therefore, one can constrain the symmetric part of $D_i y_j$, at the same time expressing $D_i u$ in terms of the derivatives of y_k:

$$\epsilon_{kij}\frac{D_j y_i + \frac{1}{2}\epsilon_{imn} y_m D_j y_n}{1+\frac{1}{4}y^2} = D_k u,$$
$$D_i y_j + D_j y_i + \tfrac{1}{2}\epsilon_{imn} y_m D_j y_n + \tfrac{1}{2}\epsilon_{jmn} y_m D_i y_n = \delta_{ij}\left(\tfrac{2}{3} D_m y_m - \tfrac{1}{3}\epsilon_{mnp} y_m\, D_n y_p\right). \tag{3.13}$$

Relations (3.13) are the irreducibility condition of the multiplet $(4,4,0)$ with the physical bosons being the first components of u and y_i, while four fermions being $D_i u$ and $D_m y_m$[1]. To prove this statement, it is sufficient to check that $D_i D_j y_k$ and $D_i D_j u$ can be expressed in terms of the u, y_i superfields and their derivatives.

First, calculating $D_i D_j y_k$ from (3.12), one can relate it to $D_i\widetilde{\Sigma}$ and $\epsilon_{jpq} D_p D_q u$:

$$D_i D_j y_k = D_i N_{km}\Lambda_{mj} + N_{km} D_i \Lambda_{mj} = \left(\frac{1}{2}\epsilon_{kmq} N_{qp} + \frac{1}{4} y_m N_{kp} + \frac{1}{4} y_k N_{mp}\right)\Lambda_{pi}\Lambda_{mj} \; +$$
$$+\; N_{km}\left(D_i\widetilde{\Sigma}\delta_{mj} + \frac{\mathrm{i}}{2}\epsilon_{ijm}\dot{u} - \frac{1}{4}\delta_{im}\epsilon_{jpq} D_p D_q u + \frac{1}{4}\delta_{ij}\epsilon_{mpq} D_p D_q u\right). \tag{3.14}$$

On the other hand, $D_i D_j y_k + D_j D_i y_k = \{D_i, D_j\}\, y_k = -2\mathrm{i}\delta_{ij}\dot{y}_k$. Comparing this with (3.14), one can find that

$$\epsilon_{ipq} D_p D_q u = -4\mathrm{i}\left(N^{-1}\right)_{ij}\dot{y}_j - 2\widetilde{\Sigma} D_i u, \quad D_i\widetilde{\Sigma} = -\mathrm{i}\left(N^{-1}\right)_{ij}\dot{y}_j + \frac{1}{8}\epsilon_{ipq} D_p u\, D_q u. \tag{3.15}$$

Next, expression for the second derivative of y_k can be found by substituting (3.15) into (3.14). Thus, we demonstrate that the second covariant derivatives of u and y_k can be expressed in terms of physical components and their time derivatives.

To construct an superconformally invariant action, one should study explicitly the $OSp(3|2)$ transformations. As special conformal transformations can be produced as commutators of the superconformal ones, it is sufficient to find transformations induced by $g_S = e^{\eta_i \mathcal{S}_i}$:

$$\delta_S t = -\mathrm{i}\eta_i\theta_i t, \;\; \delta_S\theta_i = -\eta_i t - \mathrm{i}\eta_k\theta_k\theta_i, \;\; \delta_S u = -2\mathrm{i}\eta_k\theta_k, \;\; \delta_S M_{ij} = \mathrm{i}M_{in}\left(\eta_n\theta_j - \eta_j\theta_n\right),$$
$$\delta_S z = \mathrm{i}\eta_k\lambda_k + 2\mathrm{i}\eta_k\theta_k z, \;\; \delta_S\lambda_i = \eta_i + \mathrm{i}\eta_k\theta_k\lambda_i - \mathrm{i}\eta_k\lambda_k\theta_i - \mathrm{i}\eta_i\theta_k\lambda_k. \tag{3.16}$$

The conformally invariant measure on the superspace can be constructed if one notes that

$$dt' d^3\theta' = \mathrm{Ber}\frac{\partial(t',\theta')}{\partial(t,\theta)} dt d^3\theta = dt d^3\theta\left(1 + \mathrm{i}\eta_i\theta_i\right), \tag{3.17}$$

[1] In what follows, it is slightly preferable to use $\widetilde{\Sigma}$ as the fourth fermion instead of $D_m y_m$.

and, therefore, $dtd^3\theta e^{\frac{u}{2}}$ is invariant. The action should involve another conformally invariant fermionic superfield; a suitable candidate is thus [2]

$$-12S_{\mathcal{N}=3} = \int dtd^3\theta e^{\frac{u}{2}}\Sigma = \int dtd^3\theta e^{u}\widetilde{\Sigma}. \tag{3.18}$$

Straightforward but lengthy calculation of the integral over θ_i using (3.12), (3.15) results in

$$S_{\mathcal{N}=3} = \frac{1}{2}\int dte^{u}\left(\frac{1}{4}\dot{u}^2 + g_{ij}\dot{y}_i\dot{y}_j + \right. \\ \left. + \mathrm{i}\lambda_i\dot{\lambda}_i - \mathrm{i}\widetilde{\Sigma}\dot{\widetilde{\Sigma}} + \left(-\mathrm{i}\epsilon_{ijk}\lambda_j\lambda_k + 2\widetilde{\Sigma}\lambda_i\right)\left(N^{-1}\right)_{im}\dot{y}_m - \mathrm{i}\widetilde{\Sigma}\epsilon_{ijk}\lambda_i\lambda_j\lambda_k\right), \tag{3.19}$$

where we used the same notation for superfields and their first components. The metric in the internal sector is

$$g_{ij} = \left(N^{-1}\right)_{ki}\left(N^{-1}\right)_{kj} = \frac{\delta_{ij}}{1+\frac{1}{4}y^2} - \frac{1}{4}\frac{y_iy_j}{\left(1+\frac{1}{4}y^2\right)^2}. \tag{3.20}$$

Due to $SO(3)$ symmetry, one can expect the metric to describe a 3-sphere. This becomes clear after field redefinition:

$$y_i = \frac{z_i}{1-\frac{1}{16}z^2}, \quad u = 2\log r, \quad \lambda_i = \frac{1}{r}\psi_i, \quad \widetilde{\Sigma} = -\frac{\mathrm{i}}{r}\chi, \\ \tilde{N}_{ij} = \left(1-\frac{1}{16}z^2\right)\delta_{ij} + \frac{1}{8}z_iz_j + \frac{1}{2}\epsilon_{ijm}z_m \quad \Rightarrow \\ S_{\mathcal{N}=3} = \frac{1}{2}\int dt\left(\dot{r}^2 + r^2\frac{\dot{z}_i\dot{z}_i}{\left(1+\frac{1}{16}z^2\right)^2} - \mathrm{i}\dot{\psi}_i\psi_i - \mathrm{i}\dot{\chi}\chi - \right. \\ \left. - \ \mathrm{i}\left(\epsilon_{ijk}\psi_j\psi_k + 2\chi\psi_i\right)\left(\tilde{N}^{-1}\right)_{im}\dot{z}_m - r^{-2}\chi\epsilon_{ijk}\psi_i\psi_j\psi_k\right). \tag{3.21}$$

The respective Hamiltonian, momenta and Dirac brackets read

$$H = \frac{p_r^2}{2} + \frac{1}{2r^2}\left(1+\frac{1}{16}z^2\right)^2 p_ip_i + \frac{\mathrm{i}}{2r^2}p_i\tilde{N}_{ij}\left(2\chi\psi_j + \epsilon_{jmn}\psi_m\psi_n\right), \\ p_r = \dot{r}, p_i = r^2\left(1+\frac{1}{16}z^2\right)^{-2}\dot{z}_i - \mathrm{i}\chi\psi_k\left(\tilde{N}^{-1}\right)_{ki} - \frac{\mathrm{i}}{2}\epsilon_{mnp}\psi_n\psi_p\left(\tilde{N}^{-1}\right)_{mi}, \\ \{p_r, r\} = 1, \quad \{p_i, z_j\} = \delta_{ij}, \quad \{\psi_i, \psi_j\} = \mathrm{i}\delta_{ij}, \quad \{\chi, \chi\} = \mathrm{i}. \tag{3.22}$$

The transformation laws of the components are remarkably simple:

$$\delta f|_{\theta\to 0} = \left(\epsilon_i D_i f\right)|_{\theta\to 0} \quad \Rightarrow \quad \delta r = \mathrm{i}\epsilon_i\psi_i, \quad \delta\psi_i = -\epsilon_i p_r - \frac{\epsilon_j\epsilon_{jik}\tilde{N}_{nk}p_n}{r}, \\ \delta z_i = \frac{\mathrm{i}\epsilon_j}{r}\left(-\chi\tilde{N}_{ij} + \tilde{N}_{ik}\epsilon_{kjm}\psi_m\right), \quad \delta\chi = \frac{1}{r}\epsilon_i\tilde{N}_{ji}p_j. \tag{3.23}$$

The transformation laws of the fermions (3.23) contain p_i and z_i only as a part of combination $J_i = -\tilde{N}_{ji}p_j$. As can be checked, these currents form the $so(3)$ algebra

$$\{J_i, J_j\} = \epsilon_{ijk}J_k. \tag{3.24}$$

[2] It is worth noting that this action is just the $\mathcal{N}=3$ Schwarzian action where the variables of integration were changed from “invariant” $\tau, \tilde{\theta}_i$ to $t, \theta_i \sim \xi_i(\tau, \tilde{\theta})$.

The transformations (3.23) can be reproduced via the Dirac brackets

$$\delta f = \mathrm{i}\left\{\epsilon_i Q_i, f\right\}, \quad Q_i = p_r\psi_i + \frac{1}{r}J_i\chi + \frac{1}{r}\epsilon_{ijk}J_j\psi_k. \tag{3.25}$$

These supercharges coincide with the ones constructed in Section 2 for a particular choice of J_i.

4. New variant of $\mathcal{N}=4$ superconformal mechanics

Among the constructed superconformal mechanics, the different variants with $\mathcal{N}=4$ supersymmetry have definitely attracted the most attention and are the most studied to date [7–22].

The most general $\mathcal{N}=4, d=1$ superconformal group is the exceptional supergroup $D(1,2;\alpha)$ [24, 25]. The realizations of $D(1,2;\alpha)$ in the models of supersymmetric mechanics were a subject of many papers (see, e.g., [9] and references therein). However, most of the realizations were based on one or another fixed type of the irreducible $\mathcal{N}=4, d=1$ supermultiplet. Only recently, the study of superconformal systems including some pairs of such multiplets was initiated in [14] and then continued in [18]. However, even in these generalizations, only one $su(2)$ subalgebra of R-symmetry has a bosonic realization. Second $su(2)$ algebra was realized purely on fermions. In this section, we construct supercharges and Hamiltonian of $\mathcal{N}=4$ superconformal mechanics with both $su(2)$ subalgebras having the bosonic core.

4.1. Basic ingredients

To construct $\mathcal{N}=4$ superconformal mechanics with $D(1,2;\alpha)$ dynamical supersymmetry, let us introduce the following set of bosonic and fermionic fields:

- The bosonic field r and the corresponding momentum p_r, to realize the conformal $sl(2,\mathbb{R})$ symmetry as in (2.2).
- Two triplets of the bosonic currents J^{ab} and T^{ij} obeying the Poisson brackets

$$\left\{J^{ab}, J^{cd}\right\} = -\left(\epsilon^{ac}J^{bd} + \epsilon^{bd}J^{ac}\right), \quad \left\{T^{ij}, T^{kl}\right\} = -\left(\epsilon^{ik}T^{jl} + \epsilon^{jl}T^{ik}\right). \tag{4.1}$$

  We do not fix the parameterization of these $su(2)$ subalgebras.
- Eight fermionic fields ψ^{ia} and σ^{ia} obeying the brackets

$$\left\{\psi^{ia}, \psi^{jb}\right\} = 2\mathrm{i}\,\epsilon^{ij}\epsilon^{ab}, \quad \left\{\sigma^{ia}, \sigma^{jb}\right\} = 2\mathrm{i}\,\epsilon^{ij}\epsilon^{ab}. \tag{4.2}$$

  From these fermions one can construct four $su(2)$ algebras with the generators

$$\widehat{J}^{ab}_\psi = \frac{\mathrm{i}}{4}\psi^{ia}\psi^b_i, \quad \widehat{J}^{ab}_\sigma = \frac{\mathrm{i}}{4}\sigma^{ia}\sigma^b_i, \quad \widehat{T}^{ij}_\psi = \frac{\mathrm{i}}{4}\psi^{ia}\psi^j_a, \quad \widehat{T}^{ij}_\sigma = \frac{\mathrm{i}}{4}\sigma^{ia}\sigma^j_a. \tag{4.3}$$

  All these currents obey the same brackets as in (4.1).
- Additional currents constructed from the fermionic fields ψ^{ia} and σ^{ia}

$$\widehat{V}^{ab} = \frac{\mathrm{i}}{8}\left(\psi^{ia}\sigma^b_i + \psi^{ib}\sigma^a_i\right), \quad \widehat{W}^{ij} = \frac{\mathrm{i}}{8}\left(\psi^{ia}\sigma^j_a + \psi^{ja}\sigma^i_a\right) \tag{4.4}$$

  span two $so(4)$ algebras together with the generators $\widehat{J}^{ab}_\psi + \widehat{J}^{ab}_\sigma$ and $\widehat{T}^{ij}_\psi + \widehat{T}^{ij}_\sigma$, respectively:

$$\begin{aligned}\left\{\widehat{V}^{ab}, \widehat{V}^{cd}\right\} &= -\frac{1}{4}\left(\epsilon^{ac}(\widehat{J}^{bd}_\psi + \widehat{J}^{bd}_\sigma) + \epsilon^{bd}(\widehat{J}^{ac}_\psi + \widehat{J}^{ac}_\sigma)\right),\\ \left\{\widehat{W}^{ij}, \widehat{W}^{kl}\right\} &= -\frac{1}{4}\left(\epsilon^{ik}(\widehat{T}^{jl}_\psi + \widehat{T}^{jl}_\sigma) + \epsilon^{jl}(\widehat{T}^{ik}_\psi + \widehat{T}^{ik}_\sigma)\right).\end{aligned} \tag{4.5}$$

- The final $U(1)$ current is

$$\widehat{Z} = \frac{i}{4}\psi^{ia}\sigma_{ia}. \tag{4.6}$$

Coming back to a possible superfield description of our set of fields, note that the fields $\{r, \psi^{ia}, J_{ab}\}$ fit into the $\mathcal{N} = 4$ superfield $(1,4,3)$. The rest of the fields form, probably, the spin 1 superfield $(3,4,1)$ $\{T^{ij}, \dot{\sigma}^{ia}, \mathcal{A}\}$ with $\mathcal{A}$ being auxiliary component which is invisible within the Hamiltonian description. All together, these components fit well into the unrestricted $\mathcal{N} = 4$ superfield r.

4.2. Supercharges and Hamiltonian

Using the Anzatz (2.7), the supercharges of $\mathcal{N} = 4$ superconformal mechanics can be found to be

$$\begin{aligned} Q^{ia} = {}& p_r\,\psi^{ia} - \frac{1}{r}\,2\alpha\left(T^{ij} + \frac{1}{3}\widehat{T}^{ij}_{\psi} + \widehat{T}^{ij}_{\sigma}\right)\psi^a_j + \frac{1}{r}2\alpha\left(T^{ij} + \frac{2}{3}\widehat{T}^{ij}_{\sigma}\right)\sigma^a_j + \\ &+ \frac{1}{r}\,2(1+\alpha)\left(J^{ab} + \frac{1}{3}\widehat{J}^{ab}_{\psi} + \widehat{J}^{ab}_{\sigma}\right)\psi^i_b + \frac{1}{r}\,2(1+\alpha)\left(J^{ab} + \frac{2}{3}\widehat{J}^{ab}_{\sigma}\right)\sigma^i_b + \frac{M}{r}\sigma^{ia}. \end{aligned} \tag{4.7}$$

They obey the standard brackets

$$\left\{Q^{ia}, Q^{jb}\right\} = 4\,\mathrm{i}\,\epsilon^{ij}\epsilon^{ab}H, \tag{4.8}$$

where the Hamiltonian H reads

$$\begin{aligned} H = {}& \frac{1}{2}p_r^2 + \frac{1}{r^2}\left[2(1+\alpha)^2 J^{ab}J_{ab} + 2\alpha^2 T^{ij}T_{ij} - 2(1+\alpha)J^{ab}\left((\widehat{J}_{\psi})_{ab} - (\widehat{J}_{\sigma})_{ab} + 2\widehat{V}_{ab}\right)\right] + \\ &+ \frac{1}{r^2}\left[2\alpha T^{ij}\left((\widehat{T}_{\psi})_{ij} - (\widehat{T}_{\sigma})_{ij} - 2\widehat{W}_{ij}\right) - \right. \\ &\left. - \frac{1+\alpha}{3}\widehat{J}^{ab}_{\psi}\left((\widehat{J}_{\psi})_{ab} + 6(\widehat{J}_{\sigma})_{ab}\right) + \frac{\alpha}{3}\widehat{T}^{ij}_{\psi}\left((\widehat{T}_{\psi})_{ij} + 6(\widehat{T}_{\sigma})_{ij}\right)\right] + \\ &+ \frac{1}{r^2}\left[(1+\alpha)\widehat{J}^{ab}_{\sigma}\left((\widehat{J}_{\sigma})_{ab} - \frac{8}{3}\widehat{V}_{ab}\right) - \alpha\widehat{T}^{ij}_{\sigma}\left((\widehat{T}_{\sigma})_{ij} + \frac{8}{3}\widehat{W}_{ij}\right) + 2M\widehat{Z} + \frac{M^2}{2}\right]. \end{aligned} \tag{4.9}$$

To visualize the dynamical symmetry of the system, one has to calculate the Poisson brackets between the Poincaré (4.7) and conformal supersymmetry generators S^{ia} :

$$S^{ia} = r\psi^{ia}, \tag{4.10}$$

$$\left\{S^{ia}, Q^{jb}\right\} = 4\mathrm{i}\,\epsilon^{ij}\epsilon^{ab}D + 4\,\mathrm{i}\alpha\epsilon^{ab}\left(T^{ij} + \widehat{T}^{ij}_{\psi} + \widehat{T}^{ij}_{\sigma}\right) - 4\,\mathrm{i}\,(1+\alpha)\epsilon^{ij}\left(J^{ab} + \widehat{J}^{ab}_{\psi} + \widehat{J}^{ij}_{\sigma}\right). \tag{4.11}$$

Thus, we conclude that our system possesses $D(1,2;\alpha)$ dynamical symmetry. The full R-symmetry is $sl_2 \times S^3 \times S^3$.

Let us also note that one can remove bosonic currents either J^{ab} or T^{ij} (or both of them) from the supercharges (4.7) and check that the $D(1,2;\alpha)$ algebra is still closed. The resulting systems will, however, contain eight fermions, and cubic terms with σ^{ia} will be present in the supercharge, ensuring that σ^{ia} can not be absorbed into the remaining bosonic current. Therefore, this way one obtains a different realization of $D(1,2;\alpha)$ compared to (2.15) and [9, 18, 20].

5. Conclusion

In this paper, we constructed superconformal mechanics with $\mathcal{N} = 3$ and $\mathcal{N} = 4$ supersymmetry. The important point in our construction is the analogy between the superconformal and Schwarzian mechanics [6, 13], which includes similarities in the group element, the superfield constraints and the superfield action. The insight into the number and properties of the fermionic fields gained from this analogy allowed us to construct $\mathcal{N} = 3$ and $\mathcal{N} = 4$ systems with maximal superconformal symmetries, $osp(3|2)$ with four fermions and $D(1, 2; \alpha)$ with eight fermions, respectively. Remarkably, parameterization of the $so(3)$ currents in these systems is not fixed, which allows one to consider systems with different underlying geometries.

Our study opens up possibilities of constructing new superconformal mechanical systems, especially the ones with extended $\mathcal{N} = 5, 6$ supersymmetries, which could be analogues of the generalized Schwarzians [29].

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Conflicts of interest

The authors declare no conflicts of interest.

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